Electromagnetic Forces in Continuous Media

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WARNING: This talk uses Gauss system of base dimensions!

$$\epsilon_0 = \mu_0 = 1$$

The Controversies

- Abraham-Minkowski dilemma
- Conventional Lorentz vs Einstein-Laub force densities
- Spatial distribution of force density

The Root Difficulties

- Ambiguities in classical electrodynamics (free vs bound currents, displacement current, etc.)
- Lorentz force does not contain explicitly magnetic moments
- Stability of matter is not consistent with classical electrodynamics (non-electromagnetic forces are required)

1. DENSITY OF MOMENTUM IN A MEDIUM (Abraham-Minkowski Dilemma)

Minkowski

$$\mathbf{g} = \frac{1}{4\pi c}\,\mathbf{D}\times\mathbf{B}$$

Abraham

$$\mathbf{g} = \frac{1}{4\pi c}\,\mathbf{E}\times\mathbf{H}$$

Let the field be of the following form, and completely confined to a medium:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}\,\psi(nz/c-t)\ , \qquad \mathbf{H}(z,t) = \hat{\mathbf{y}}\,Z\,\psi(nz/c-t)$$

$$n = \sqrt{\epsilon\mu}\ , \qquad Z = \sqrt{\epsilon/\mu}$$

(no absorption or dispersion for now)

Let the density of energy be given by the conventional formula

$$U(z,t) = \frac{1}{8\pi}\left[\epsilon\,E^2(z,t) + \mu\,H^2(z,t)\right]\ , \qquad \mathcal{E} = \int_{-\infty}^{\infty} U(z,t)\,\mathrm{d}z = \mathrm{const} < \infty$$

total pulse energy per unit surface

$$\mathcal{P} = \int_{-\infty}^{\infty} g_z(z,t)\,\mathrm{d}z = \frac{n\,\mathcal{E}}{c}$$

Minkowski's momentum of the EM pulse per unit surface

$$\mathcal{P} = \int_{-\infty}^{\infty} g_z(z,t)\,\mathrm{d}z = \frac{\mathcal{E}}{n\,c}$$

Abraham's momentum of the EM pulse per unit surface

MOMENTUM OF A PHOTON

Minkowski

$$p = \frac{n\,E}{c}$$

Abraham

$$p = \frac{E}{n\,c}$$

In vacuum:

$$E = \hbar\,\omega\ , \qquad p = \hbar\,k\ , \qquad k = \omega/c\ , \qquad p = E/c$$

Generalize the wave number:

$$p = \hbar\,k$$

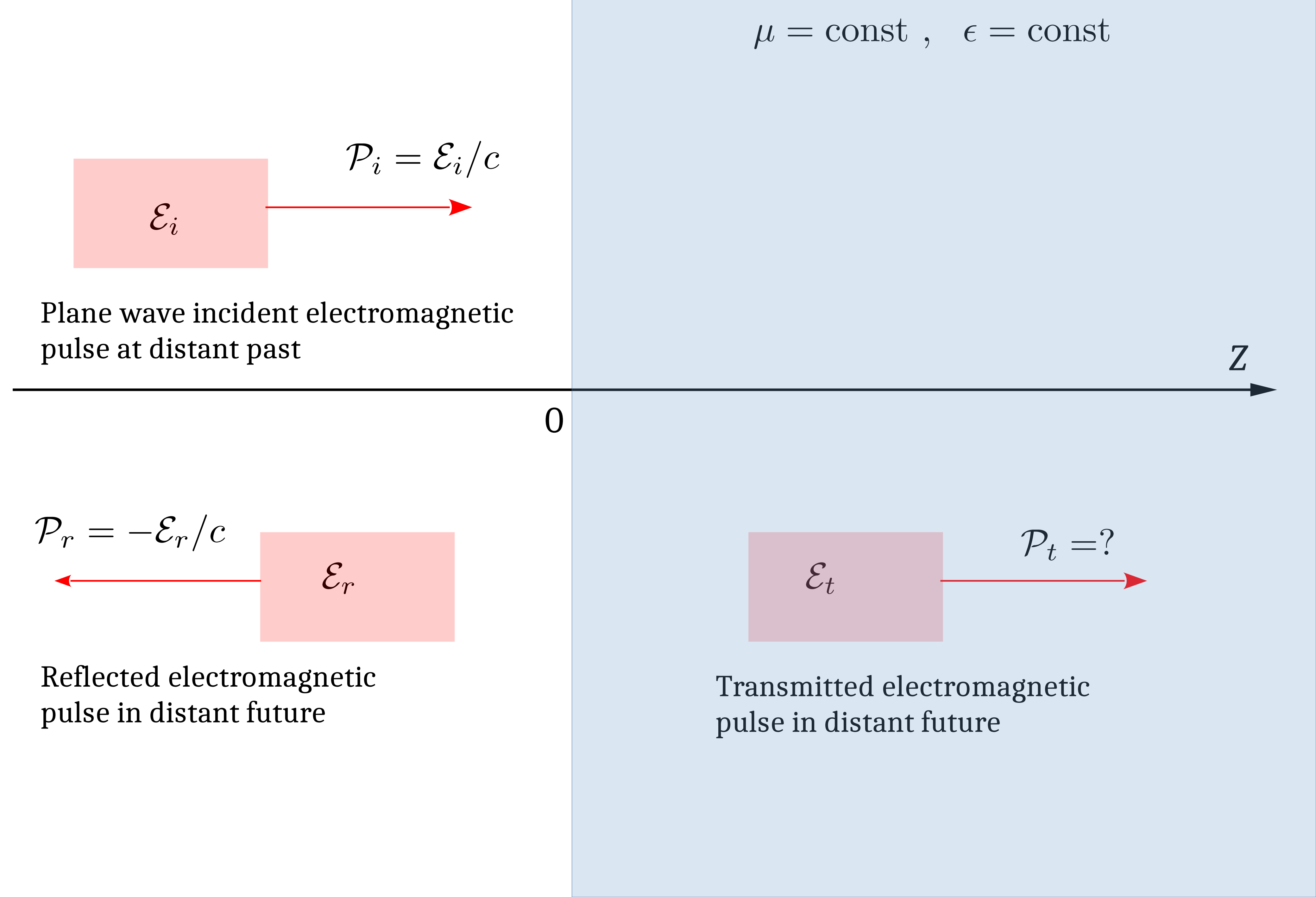
$$k = n\,\omega/c \quad \Rightarrow \quad p = n\,E/c$$

Generalize the phase velocity:

$$p = m_{\mathtt{rel}}\,v_{\mathtt{ph}}\ , \quad m_{\mathtt{rel}} = \hbar\omega/c^2\ , \quad v_{\mathtt{ph}} = c$$

$$v_{\mathtt{ph}} = c/n \quad \Rightarrow \quad p = E/nc$$

EXAMPLE 1: TRANSIENT PULSE ENTERING A SEMI-INFINITE MEDIUM



EXAMPLE 1 (cont.)

$z < 0:$

incident field:

$$\begin{cases} \mathbf{E}_i(z,t) = \hat{\mathbf{x}}\,\psi(t - z/c) \\ \mathbf{H}_i(z,t) = \hat{\mathbf{y}}\,\psi(t - z/c) \end{cases}$$

reflected field:

$$\begin{cases} \mathbf{E}_r(z,t) = \quad \hat{\mathbf{x}}\,A_r\,\psi(t + z/c) \\ \mathbf{H}_r(z,t) = -\hat{\mathbf{y}}\,A_r\,\psi(t + z/c) \end{cases}$$

$z > 0:$

transmitted field:

$$\begin{cases} \mathbf{E}_t(z,t) = \quad \hat{\mathbf{x}}\,A_t\,\psi(t - nz/c) \\ \mathbf{H}_t(z,t) = \hat{\mathbf{y}}\,Z\,A_t\,\psi(t - nz/c) \end{cases}$$

$$A_r = \frac{Z-1}{Z+1}\,, \qquad A_t = \frac{2}{Z+1}$$

$n = \sqrt{\epsilon\mu}$ (index of refraction)

$Z = \sqrt{\epsilon/\mu}$ (impedance)

$$\mathcal{E}_i = \frac{c}{4\pi}\int_{-\infty}^{\infty} \psi^2(t)\,\mathrm{d}t$$

$$\mathcal{P}_i = \mathcal{E}_i/c$$

$$\mathcal{E}_r = A_r^2\,\mathcal{E}_i$$

$$\mathcal{P}_r = -\mathcal{E}_r/c$$

$$\mathcal{E}_t = A_t^2\,Z\,\mathcal{E}_i$$

$$\mathcal{P}_t = ?$$

EXAMPLE 1 (cont.)

Let us use momentum conservation:

$$\mathcal{P}_i = \mathcal{P}_r + \mathcal{P}_t + \mathcal{P}_b \;\Rightarrow\; \mathcal{P}_t = \mathcal{P}_i - \mathcal{P}_r - \mathcal{P}_b$$

incident ($\mathcal{P}_i$), reflected ($\mathcal{P}_r$), transmitted ($\mathcal{P}_t$), block (mechanical) ($\mathcal{P}_b$)

We want to use this equation to find the momentum of the transmitted pulse. But we need to know the mechanical momentum of the block.

$$\mathcal{P}_b = \int_{-\infty}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}z \; f_z(z,t)$$

density of force acting on the block (*z*-projection)

Force on a point charge *q* moving according to the law $\mathbf{r}_q(t)$ with the velocity $\mathbf{v}_q(t)=\mathrm{d}\mathbf{r}_q(t)/\mathrm{d}t$:

$$\mathbf{F}(t) = q\,\mathbf{E}(\mathbf{r}_q(t),t) + \frac{q}{c}\,\mathbf{v}_q(t) \times \mathbf{B}(\mathbf{r}_q(t),t)$$

Force density on a continuous distribution of charge and current:

$$\mathbf{f}(\mathbf{r},t) = \rho(\mathbf{r},t)\mathbf{E}(\mathbf{r},t) + \frac{1}{c}\,\mathbf{J}(\mathbf{r},t) \times \mathbf{B}(\mathbf{r},t)$$

EXAMPLE 1 IN A NON-MAGNETIC MEDIUM (cont.)

In a non-magnetic medium, and for the geometry of Example 1:

$$\mathbf{B}(\mathbf{r},t) = \mathbf{H}(\mathbf{r},t)$$
$$\rho(\mathbf{r},t) = 0$$
$$\mathbf{J}(\mathbf{r},t) = \frac{\partial \mathbf{P}(\mathbf{r},t)}{\partial t}$$
$$\mathbf{P}(\mathbf{r},t) = \frac{\epsilon - 1}{4\pi}\mathbf{E}(\mathbf{r},t)$$

From the solution to Maxwell's equations from the previous page::

$$f_z(\mathbf{r},t) = \frac{\epsilon - 1}{4\pi c} A_t^2\, Z\, \psi'\left(t - \frac{nz}{c}\right)\psi\left(t - \frac{nz}{c}\right)$$

$$\mathcal{P}_b = \int_{-\infty}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}z\ f_z(z,t) = 2\mathcal{P}_i \frac{n-1}{n+1}$$

$$\mathcal{P}_t = \mathcal{P}_i - \mathcal{P}_r - \mathcal{P}_b = \frac{4\,\mathcal{P}_i}{(n+1)^2}$$

momentum of the transmitted pulse

Minkowski and Abraham predictions:

$$\mathcal{E}_t = A_t^2\, Z\, \mathcal{E}_i \quad \Rightarrow \quad \mathcal{P}_t = \begin{cases} \dfrac{n\mathcal{E}_t}{c} = \dfrac{4n^2\,\mathcal{P}_i}{(n+1)^2}\,, & \text{Minkowski} \\ \dfrac{\mathcal{E}_t}{nc} = \dfrac{4\,\mathcal{P}_i}{(n+1)^2}\,, & \text{Abraham} \end{cases}$$

Looks like Abraham was correct

ABRAHAM FORMULA IS CORRECT IN ANY NON-MAGNETIC MEDIUM

In a closed system “block + field”:

$$\left.\begin{array}{l} \mathbf{p}_{\texttt{block}}(t) + \mathbf{p}_{\texttt{field}}(t) = \text{const} \\ \dfrac{\partial \mathbf{p}_{\texttt{block}}(t)}{\partial t} = \mathbf{F}(t) \\ \mathbf{p}_{\texttt{field}}(t) = \int \mathbf{g}(\mathbf{r},t)\,\mathrm{d}^3 r \end{array}\right\} \Rightarrow \mathbf{F}(t) = -\frac{\partial}{\partial t}\int \mathbf{g}(\mathbf{r},t)\,\mathrm{d}^3 r$$

Using Abraham’s formula

$$\mathbf{F}(t) = -\frac{1}{4\pi c}\frac{\partial}{\partial t}\int \mathbf{E}(\mathbf{r},t)\times\mathbf{H}(\mathbf{r},t)\,\mathrm{d}^3 r$$

Integration over the entire space!

We will now use Maxwell equations in non-magnetic media to show that the force defined above is just the total Lorentz force acting on the block

$$\begin{cases} \nabla\times\mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{D}}{\partial t} \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} \end{cases} \Leftrightarrow \begin{cases} \nabla\times\mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{4\pi}{c}\mathbf{J} \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} \end{cases} \Leftarrow \begin{cases} \mathbf{J} = \dfrac{\partial \mathbf{P}}{\partial t} \\ \mathbf{D} = \mathbf{E} + 4\pi\mathbf{P} \end{cases}$$

where

ABRAHAM FORMULA IS CORRECT IN ANY NON-MAGNETIC MEDIUM (cont.)

$$\mathbf{F} = -\frac{1}{4\pi c}\frac{\partial}{\partial t}\int \mathbf{E}\times\mathbf{H}\,\mathrm{d}^3r = -\frac{1}{4\pi c}\int\left[\frac{\partial \mathbf{E}}{\partial t}\times\mathbf{H} + \mathbf{E}\times\frac{\partial \mathbf{H}}{\partial t}\right]\mathrm{d}^3r$$

$$= -\frac{1}{4\pi c}\int\left[(c\nabla\times\mathbf{H} - 4\pi\mathbf{J})\times\mathbf{H} - c\,\mathbf{E}\times(\nabla\times\mathbf{E})\right]\mathrm{d}^3r$$

$$= \frac{1}{c}\int \mathbf{J}\times\mathbf{H}\,\mathrm{d}^3r + \frac{1}{4\pi}\int\left[\mathbf{H}\times(\nabla\times\mathbf{H}) + \mathbf{E}\times(\nabla\times\mathbf{E})\right]\mathrm{d}^3r$$

because the medium is non-magnetic

$$= \frac{1}{c}\int \mathbf{J}\times\mathbf{B}\,\mathrm{d}^3r + \int \rho\,\mathbf{E}\,\mathrm{d}^3r$$

This is the set of Maxwell's equations we have used

$$\begin{cases} \nabla\times\mathbf{H} = \dfrac{1}{c}\dfrac{\partial \mathbf{E}}{\partial t} + \dfrac{4\pi}{c}\mathbf{J} \\ \nabla\times\mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{H}}{\partial t} \end{cases}$$

Derivation of the second term relied on these identities:

$$\mathbf{a}\times(\nabla\times\mathbf{a}) = (1/2)\nabla(\mathbf{a}\cdot\mathbf{a}) - (\mathbf{a}\cdot\nabla)\mathbf{a}$$

$$\int[(\mathbf{a}\cdot\nabla)\mathbf{a} + (\nabla\cdot\mathbf{a})\mathbf{a}]\mathrm{d}^3r = 0$$

SO WHY THE ABRAHAM-MINKOWSKI "CONTROVERSY" DOESN'T GO AWAY?

Two main reasons:

1) Wrong definitions of the current and the "free current" conundrum

2) Incorrect or confusing treatment of magnetic forces and currents

But, for now, we look at non-magnetic media. So it's mainly Reason 1.

In this paper:
"**Remarks on forces and the energy-momentum tensor in macroscopic electrodynamics**", V.L.Ginzburg, V.A.Ugarov, UFN **118**, 175 (1976)
authors argue at length that the polarization current $\partial \mathbf{P}/\partial t$ is just an "ordinary current" and that magnetic field should act on it just as it would on the conductivity current or what is often known as "free current".

But no.

2. FORCES IN MAGNETIC MEDIA (LORENTZ vs EINSTEIN-LAUB)

* Everything we know about electromagnetic forces, momentum and energy comes from the formula for the microscopic Lorentz force and Maxwell's equations for the fields. There is no other experimental basis for such considerations.

* However the standard Lorentz force says nothing about forces on magnetic moments. One must make a model – magnetization is caused by electric current (standard Lorentz interpretation) OR by displacement of magnetic poles (Einstein-Laub interpretation).

* The above choice changes the form of force density acting in magnetized media.

* Actually, if we repeat the calculation with a pulse entering a transparent half-space in the presence of magnetization, the Abraham's momentum density will be inconsistent with the standard Lorentz force density.

* Many papers have suggested to use "generalized" or "Einstein-Laub" force and that standard expression is inconsistent with conservation laws.

STANDARD LORENTZ AND EINSTEIN-LAUB FORCE DENSITIES

$$\mathbf{f} = \rho_e \mathbf{E} + \frac{1}{c}\mathbf{J}_e \times \mathbf{B}$$

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M}$$

$$\rho_e = \frac{1}{4\pi}\nabla \cdot \mathbf{E}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_e = 0$$

$$\mathbf{f} = \rho_e \mathbf{E} + \rho_m \mathbf{H} + \frac{1}{c}\left(\mathbf{J}_e \times \mathbf{H} - \mathbf{J}_m \times \mathbf{E}\right)$$

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} \qquad \mathbf{J}_m = \frac{\partial \mathbf{M}}{\partial t}$$

$$\rho_e = \frac{1}{4\pi}\nabla \cdot \mathbf{E} \qquad \rho_m = \frac{1}{4\pi}\nabla \cdot \mathbf{H}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_e = 0 \qquad \frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathbf{J}_m = 0$$

Standard Lorentz force density

Generalized Lorentz or Einstein-Laub force density

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

Same set of Maxwell's equations is used to determine the fields

EXAMPLE 1 IN MAGNETIC MEDIA

With account of magnetization, the total mechanical momentum transferred to the material block in Example 1 is:

From standard Lorentz force:

$$\mathcal{P}_b = \frac{2\mathcal{P}_i}{(Z+1)^2}\frac{\epsilon-\mu}{\mu}$$

From Abraham's momentum density

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}$$

and conservation of the total momentum of the system "block + field":

$$\mathcal{P}_b = \frac{2\mathcal{P}_i}{(Z+1)^2}\frac{\epsilon+\mu-2}{\mu}$$

$$Z = \sqrt{\epsilon/\mu}$$

From Einstein-Laub force:

$$\mathcal{P}_b = \frac{2\mathcal{P}_i}{(Z+1)^2}\frac{\epsilon+\mu-2}{\mu}$$

From "my" momentum density

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}$$

and conservation of the total momentum of the system "block + field":

$$\mathcal{P}_b = \frac{2\mathcal{P}_i}{(Z+1)^2}\frac{\epsilon-\mu}{\mu}$$

Actually, standard Lorentz force is consistent with "my" momentum density in general

THERE ARE TWO INTERNALLY NON-CONTRADICTORY THEORIES

$$\mathbf{f} = \rho_e \mathbf{E} + \frac{1}{c}\mathbf{J}_e \times \mathbf{B}$$

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} + c\nabla \times \mathbf{M}$$

$$\rho_e = \frac{1}{4\pi}\nabla \cdot \mathbf{E}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J} = 0$$

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{B}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$$

Magnetization is caused by electric current. Fundamental fields are **E** and **B**

$$\mathbf{f} = \rho_e \mathbf{E} + \rho_m \mathbf{H} + \frac{1}{c}\left(\mathbf{J}_e \times \mathbf{H} - \mathbf{J}_m \times \mathbf{E}\right)$$

$$\mathbf{J}_e = \frac{\partial \mathbf{P}}{\partial t} \qquad \mathbf{J}_m = \frac{\partial \mathbf{M}}{\partial t}$$

$$\rho_e = \frac{1}{4\pi}\nabla \cdot \mathbf{E} \qquad \rho_m = \frac{1}{4\pi}\nabla \cdot \mathbf{H}$$

$$\frac{\partial \rho_e}{\partial t} + \nabla \cdot \mathbf{J}_e = 0 \qquad \frac{\partial \rho_m}{\partial t} + \nabla \cdot \mathbf{J}_m = 0$$

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E} \times \mathbf{H}$$

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{H}$$

Magnetization is caused by displacement of magnetic poles. Fundamental fields are **E** and **H**

(I will comment on the density of energy separately)

SO WHICH THEORY SHOULD WE USE ?

* Textbook electrodynamics is really based on the second model, although it twists itself into knots not to acknowledge that it assumes the existence of magnetic poles or considers **H** rather than **B** to be the fundamental field.

* There have been a lot of activity recently to prove that the second model is right by searching for contradictions in the first model. Physicists are really attached to the second model for a number of reasons!

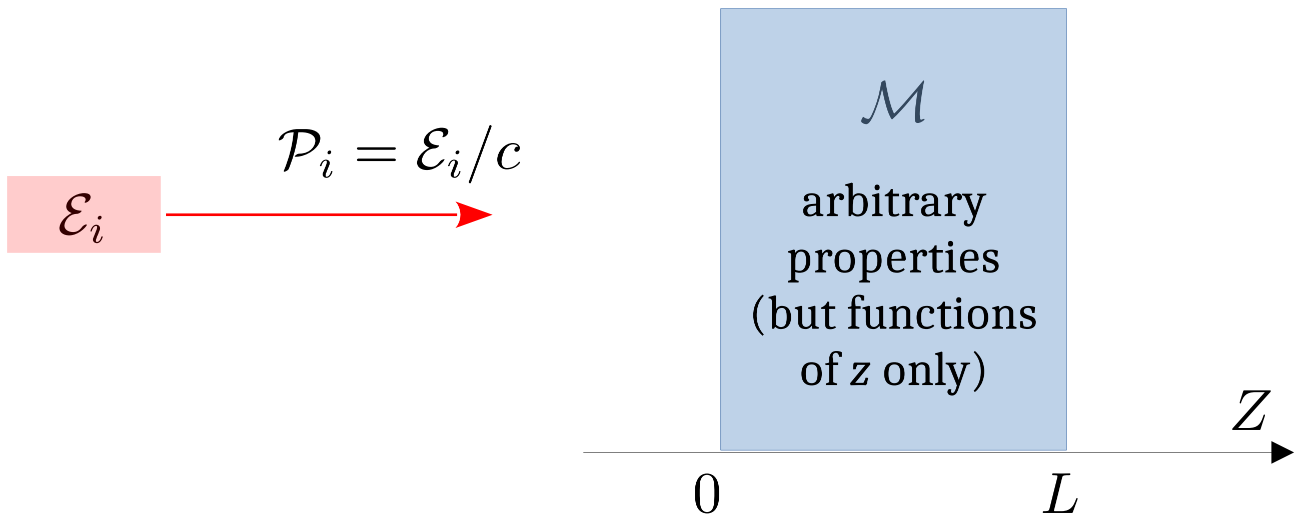
* In many cases (but not always) both theories will predict the same measurable quantity, so it is hard to find contradictions.

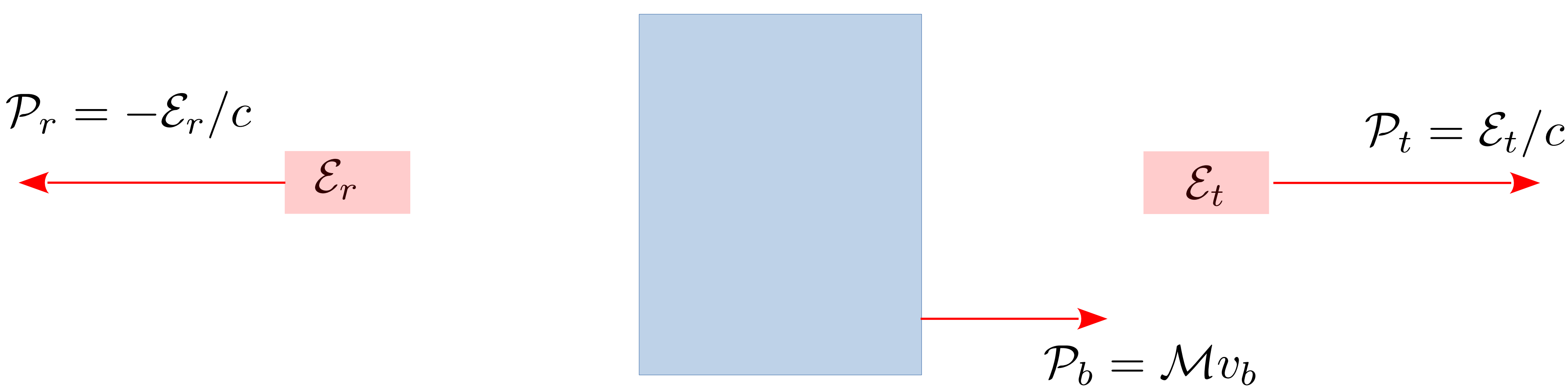
* In Example 1, the theories predict different mechanical momentum of the block. But is this quantity measurable separately from the momentum of the pulse (which is inside the block)? Perhaps, but this is very difficult to measure.

* A lot of attention was focused on the so-called Balazs thought experiment, which looks at the motion of the center of energy of the system “field+block”. This is a more subtle test than just looking at the conservation of energy and momentum.

2. L-EL dilemma

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT





EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

Fields in free space:

$$\mathbf{E}(z,t) = \hat{\mathbf{x}} \begin{cases} \psi_i\left(t - \frac{z}{c}\right) + \psi_r\left(t + \frac{z}{c}\right), & z < 0 \\ \psi_t\left(t - \frac{z-L}{c}\right), & z > L \end{cases}$$

$$\mathbf{B}(z,t) = \hat{\mathbf{y}} \begin{cases} \psi_i\left(t - \frac{z}{c}\right) - \psi_r\left(t + \frac{z}{c}\right), & z < 0 \\ \psi_t\left(t - \frac{z-L}{c}\right), & z > L \end{cases}$$

Incident pulse is arbitrary but contains a finite energy

The reflected and transmitted pulses are uniquely determined by the form of the incident pulse and the properties of the block. We do not need to know these functions in detail

All calculations are done in free space where **H**=**B**

$$\mathcal{E}_\alpha = \frac{c}{4\pi}\int_{-\infty}^{\infty} \psi_\alpha^2(t)dt\ , \quad \alpha = i, r, t\ .$$

$$\mathcal{P}_i = \mathcal{E}_i/c\ ,\ \mathcal{P}_r = -\mathcal{E}_r/x\ ,\ \ \mathcal{P}_t = \mathcal{E}_t/c\ .$$

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

(b) Energy

infinitesimally small constant

Energy absorbed within the block (either into heat or some other forms of internal energy)

$$\mathcal{Q} = \int_{-\infty}^{\infty} [S_z(-\varepsilon, t) - S_z(L+\varepsilon, t)]\,\mathrm{d}t$$

The two expressions for Poynting vector coincide in free space

$$\mathcal{Q} = \frac{c}{4\pi}\int_{-\infty}^{\infty} [E_x(-\varepsilon, t)B_y(-\varepsilon, t) - E_x(L+\varepsilon, t)B_y(L+\varepsilon, t)]\,\mathrm{d}t$$

use fields from previous page (cross-terms cancel out)

$$\mathcal{Q} = \frac{c}{4\pi}\int_{-\infty}^{\infty} [\psi_i^2(t) - \psi_r^2(t) - \psi_t^2(t)]\,\mathrm{d}t$$

$$\mathcal{Q} = \mathcal{E}_i - \mathcal{E}_r - \mathcal{E}_t$$

Energy is conserved in both theories (not a surprise)

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

(c) Momentum $\mathcal{P}_b = \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\varepsilon}^{L+\varepsilon} \mathrm{d}z \, f_z(z,t)$

In the 1D geometry considered the two competing force densities can be written identically as

Lorentz: $$\mathbf{f}_L = \frac{1}{4\pi}\left[(\nabla\times\mathbf{B})\times\mathbf{B} + (\nabla\times\mathbf{E})\times\mathbf{E} - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{B})}{\partial t}\right]$$

Einstein-Laub: $$\mathbf{f}_{EL} = \frac{1}{4\pi}\left[(\nabla\times\mathbf{H})\times\mathbf{H} + (\nabla\times\mathbf{E})\times\mathbf{E} - \frac{1}{c}\frac{\partial(\mathbf{E}\times\mathbf{H})}{\partial t}\right]$$

These terms are known as the Abraham force (it does not transfer total momentum

$$\mathcal{P}_b = -\frac{1}{8\pi}\int_{-\infty}^{\infty}\mathrm{d}t\int_{-\varepsilon}^{L+\varepsilon}\left(\frac{\partial E_x^2}{\partial z} + \frac{\partial F_y^2}{\partial z}\right)\mathrm{d}z\ , \qquad F = \begin{cases} B\ , & \text{Lorentz} \\ H\ , & \text{Einstein-Laub}\end{cases}$$

$$\mathcal{P}_b = \frac{1}{8\pi}\int_{-\infty}^{\infty}\left[E_x^2(-\varepsilon,t) + B_y^2(-\varepsilon,t) - E_x^2(L+\varepsilon,t) - B_y^2(L+\varepsilon,t)\right]\mathrm{d}t$$

$$\mathcal{P}_b = \frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\psi_{\mathtt{i}}^2(t) + \psi_{\mathtt{r}}^2(t) - \psi_{\mathtt{t}}^2(t)\right]\mathrm{d}t \quad \Rightarrow \quad \mathcal{P}_b = \mathcal{P}_i + \mathcal{P}_r - \mathcal{P}_t$$

Momentum is conserved and the same in both theories (also not a surprise)

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

(d) Center of energy motion

$$\mathcal{E}_{\text{tot}}\, z_{\text{ce}}(t) = \int_{-\infty}^{\infty} z\, u_{\text{tot}}(z)\, \mathrm{d}z$$

before the collision:

$$\mathcal{E}_{\text{tot}}\, z_{\text{ce}}(t) = \mathcal{E}_i z_i(t) + \mathcal{M}c^2\, z_0$$

after the collision:

$$\mathcal{E}_{\text{tot}}\, z_{\text{ce}}(t) = \mathcal{E}_r\, z_r(t) + \mathcal{E}_t\, z_t(t) + \mathcal{M}c^2 \left(z_0 - \Delta_b + \delta_q + v_b t\right)$$

We want to show that these expressions describe the same linear function of time

Some definitions:

Centers of energy of the incident, reflected, and transmitted pulses

$$z_i(t) = c(t - \tau_i)\ , \qquad \tau_i = \frac{c}{4\pi\mathcal{E}_i}\int \psi_i^2(t)\, t\, \mathrm{d}t$$

$$z_r(t) = -c(t - \tau_r)\ , \qquad \tau_r = \frac{c}{4\pi\mathcal{E}_r}\int \psi_r^2(t)\, t\, \mathrm{d}t$$

$$z_t(t) = c(t - \tau_t) + L\ , \qquad \tau_t = \frac{c}{4\pi\mathcal{E}_t}\int \psi_t^2(t)\, t\, \mathrm{d}t$$

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

Definitions(cont.):

$$z_b(t) = z_0 - \Delta_b + \delta_q + v_b\, t$$ ← Law of motion of the block after the collision

z_0 ← Center of mass (=energy) of the block before collision, can be 0 (cancels out)

$$v_b = \frac{1}{\mathcal{M}} \int_{-\infty}^{\infty} \mathcal{F}(t)\,\mathrm{d}t\,, \quad \Delta_b = \frac{1}{\mathcal{M}} \int_{-\infty}^{\infty} \mathcal{F}(t)\,t\,\mathrm{d}t$$

$$\delta_q = \frac{1}{\mathcal{M}c^2} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\varepsilon}^{L+\varepsilon} q(z,t)\,\mathrm{d}z$$ ← Center of energy that was transferred from the pulse to the block (typically, in the form of heat)

$$q(z,t) = -\nabla\cdot\mathbf{S}(z,t) = -\partial S_z(z,t)/\partial z$$ ← We use the appropriate expression for the Poynting vector **S** in each model

So, we need to verify the equation

$$\mathcal{E}_i c(t-\tau_i) \stackrel{?}{=} -\mathcal{E}_r c(t-\tau_r) + \mathcal{E}_t[c(t-\tau_t)+L] + \mathcal{M}c^2(\delta_q - \Delta_b + v_b t)$$

From momentum conservation, all terms proportional to time *t* cancel out

$$\mathcal{E}_i c\tau_i + \mathcal{E}_r c\tau_r + \mathcal{E}_t(L - c\tau_t) \stackrel{?}{=} \mathcal{M}c^2(\Delta_b - \delta_q)$$

EXAMPLE 2: BALAZS THOUGHT EXPERIMENT (cont.)

Calculating the shifts

We need to verify this equation

$$\mathcal{E}_i c\tau_i + \mathcal{E}_r c\tau_r + \mathcal{E}_t (L - c\tau_t) \stackrel{?}{=} \mathcal{M}c^2(\Delta_b - \delta_q)$$

$$\Delta_b = \frac{1}{\mathcal{M}} \int_{-\infty}^{\infty} \mathcal{F}(t)\, t\, \mathrm{d}t = \mathcal{E}_i c\tau_i + \mathcal{E}_r c\tau_r - \mathcal{E}_t c\tau_t + I$$

$$\delta_q = -\frac{1}{\mathcal{M}c^2} \int_{-\infty}^{\infty} \mathrm{d}t \int_{-\varepsilon}^{L+\varepsilon} \frac{\partial S_z(z,t)}{\partial z}\, z\, \mathrm{d}z = -L\mathcal{E}_t + I$$

where

$$I := \int_0^L \mathrm{d}z \int_{-\infty}^{\infty} S_z(z,t)\, \mathrm{d}t$$ (but this term cancels)

Substituting these results in the original equation we wish to verify, we find that the equality indeed holds irrespectively of which model is used.

But the two "shifts", Δ_b and δ_q, in the two models differ; only their difference is fixed.

BOTH LORENTZ AND EINSTEIN-LAUB FORCE DENSITIES ARE CONSISTENT WITH ALL THE CONSERVATION LAWS.

WHY EINSTEIN-LAUB MODEL IS OFTEN PREFERRED?

* In Lorentz model, the Poynting vector is discontinuous at the surface of a magnetized body and there are surface terms in the heatng rate density $q(\mathbf{r},t)$. Many physicists find this counter-intuitive or impossible.

* We did see that the spatial distribution of $q(\mathbf{r},t)$ is different in the two models, although the total heating rate $Q(t)$ (over a finite body) is the same.

* The Lorentz model predicts that, in a homogeneous medium with negative refraction defined as $\mathrm{Im}[\epsilon(\omega)\mu(\omega)] < 0$ and a monochromatic process, we have $q(\mathbf{r})<0$ in the bulk and $q(\mathbf{r})>0$ on the surface (still, $Q>0$). This seems to violate the second law of thermodynamics.

* Some papers object to "hidden momentum", but this is a real relativistic effect, which is in fact present in both theories.

* In general, the Einstein-Laub model is more symmetric as it treats magnetic and electric polarization on the same footing. However, it operates with magnetic charges, which have never been observed.

POYNTING THEOREM AND ENERGY DENSITY IN THE LORENTZ MODEL

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$$

$$\nabla\times\mathbf{H} = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t} \quad \Longleftrightarrow \quad \nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_e$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\cdot\mathbf{S} + \frac{\partial}{\partial t}\underbrace{\frac{\mathbf{E}\cdot\mathbf{E}+\mathbf{B}\cdot\mathbf{B}}{8\pi}}_{U_f} + \underbrace{\mathbf{J}_e\cdot\mathbf{E}}_{W} = 0$$

U_f

Density of electromagnetic energy (same as in vacuum)

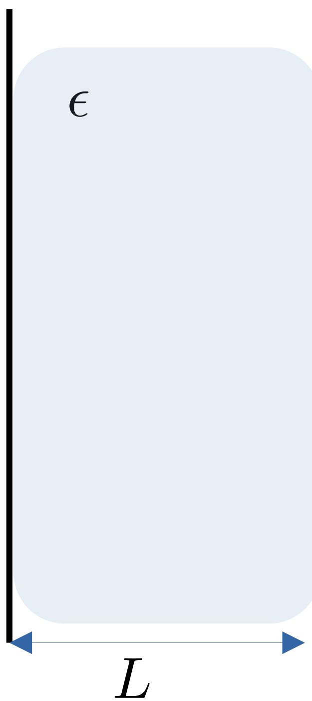
$W = q + \partial U_m/\partial t$

Density of power exerted by the electric field on the induced current. Irreversible part (q) is the density of heating rate, the rest is stored in the medium as the potential energy U_m .

POYNTING THEOREM AND ENERGY DENSITY IN THE LORENTZ MODEL (cont.)

Proverbial example of a parallel-plate capacitor

$\phi = 0$ $\qquad$ $\phi = L E_0$



$$\text{Energy Density} = \frac{\epsilon\, E_0^2}{8\pi} = \underbrace{\frac{E_0^2}{8\pi}}_{U_f} + \underbrace{\frac{(\epsilon - 1)\, E_0^2}{8\pi}}_{U_m}$$

$$U_m = \int_0^t W(t')\mathrm{d}t' = \int_0^t J_e(t')\, E(t')\mathrm{d}t'$$

$$= \int_0^t \frac{\partial P(t')}{\partial t'} E(t')\mathrm{d}t' = \frac{\epsilon - 1}{4\pi} \int_0^t \frac{\partial E(t')}{\partial t'} E(t')\mathrm{d}t'$$

$$= \frac{\epsilon - 1}{8\pi} \left[E^2(t) - E^2(0)\right]$$

(This is a reversible process if the field is changed slowly)

$$E(t) = E_0 \ , \quad E(0) = 0 \quad \Longrightarrow \quad U_m = \frac{\epsilon - 1}{8\pi} E_0^2$$

2. L-EL dilemma

PRODUCTION OF ENTROPY BY A PULSE PROPAGATING THROUGH MAGNETIC MEDIUM

before interaction

$\mathcal{E}_i$ →

during interaction

← $\mathcal{E}_r$

$$\mathcal{Q}_1 = (1-\mu)(\mathcal{E}_i - \mathcal{E}_r)$$

$$\mu(\mathcal{E}_i - \mathcal{E}_r)$$

after interaction

← $\mathcal{E}_r$

$$\mathcal{Q}_1 = (1-\mu)(\mathcal{E}_i - \mathcal{E}_r)$$

$$\mathcal{Q}_2 = (\mu - 1)(\mathcal{E}_i - \mathcal{E}_r) = -\mathcal{Q}_1$$

→

$$\mathcal{E}_t = \mathcal{E}_i - \mathcal{E}_r$$

CONCLUSIONS

* Abraham-Minkowski dilemma is not really a dilemma; it is clear that Minkowski's formula is wrong. However, this mostly concerns non-magnetic media. In magnetic media, there is a choice between two forms of the Abraham's momentum density and Poynting vector (and the corresponding force densities)

$$\mathbf{g} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{B}\ , \quad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{B} \qquad \Big| \qquad \mathbf{g} = \frac{1}{4\pi c}\mathbf{E}\times\mathbf{H}\ , \quad \mathbf{S} = \frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$$

* The two models are, in principle, experimentally distinguishable (Example 1 predicts different mechanical momenta transferred to the block; Example 2 predicts different mechanical shifts). But the effects are very small and hard to measure.

* The two models predict different densities of heating rates and probably place different restrictions on the linear constitutive coefficients. However, it is not always clear which work done by the fields is reversible and which is accompanied by production of entropy. This mainly concerns magnetization due to spin alignment (a macroscopic quantum effect). In metamaterials, magnetic effects are caused by ordinary classical currents.

* The notion of bound and free currents and charges is outdated, causes a lot of confusion, and should be abandoned. A more physically transparent classification is that of external and induced currents and charges.

CONCLUSIONS (ARGUMENTS AGAINST EINSTEIN-LAUB MODEL)

* Einstein-Laub model can be obtained starting from generalized Maxwell's equations with magnetic and electric charges. Then the notation **H** is really the magnetic field, which appears in the formula for the force density and **B** is an auxiliary quantity. But then the expressions for electric and magnetic currents in Einstein-Laub model are not of the most general form.

* Another interpretation is that really there are no magnetic poles, but different types of electric current are subject to different laws. However, there is no general way to separate the total current density **J** into $\partial \mathbf{P}/\partial t$ and $c\nabla \times \mathbf{M}$

* In the latter interpretation it is also not clear why we have **H** in the expression (an auxiliary quantity, which may not be defined) rather than **B**.