Extinction of electromagnetic waves

Vadim A. Markel

Abstract Although Beer-Lambert law has been known since eighteenth century, the phenomenon of extinction (total attenuation) of electromagnetic waves is far from trivial. At the root of all difficulties is the fact that extinction occurs as a result of interference between the incident and the scattered fields. But the measurable physical field is the sum of these two components. Historically, this led to various misconceptions and paradoxes. Although the mathematical theory of electromagnetic extinction has roots in the work of Rayleigh, Lorentz and Mie, some important questions were convincingly resolved only in the 21st century. Extinction of complex and partially-coherent fields is still an active area of research, as is the theory of measurements. In this chapter, we cover the theoretical underpinnings of extinction by bounded (but not necessarily small or spherical) particles. While we focus on extinction, absorption and scattering are also addressed. We do not discuss solutions to specific electromagnetic problems. In particular, we do not derive the Mie solution, which is amply covered elsewhere. In fact, the chapter does not contain a single special function! We also do not cover numerical methods, although some references are given. Rather, we focus on conceptual questions, general relations, and approximations that are applicable to broad classes of targets (scattering objects) and incident fields. We start with the definitions of absorbed, scattered and extinguished powers in terms of volume and surface integrals. This material is very general as we place only minimal restrictions on the form of the incident field and properties of the target. Next, we consider in detail monochromatic fields, derive the optical theorem and its generalizations, and state formal algebraic definitions of absorbed, scattered and extinguished powers. Next we discuss in substantial detail non-monochromatic partially-coherent fields. The chapter is concluded with the theory of measurements (operational definition of extinction) and resolution of related paradoxes, including the classical extinction paradox.

University of Pennsylvania, e-mail: vmarkel@upenn.edu

Contents

1 Introduction

Electromagnetic scattering theory has been amply covered in the literature. In addition to the classical exposition by R. Newton [1], the well-known book by Bohren and Huffman [2], a series of books by Mishchenko, Travis and Lacis [3, 4], the collected works [5], we can mention the recent two-part tutorial by Frezza, Mangini and Tedeschi [6, 7]. This chapter is different in several respects, as we do not consider any particular shape of the target (the ubiquitous Mie solution is not covered) or numerical approaches, but focus instead on conceptual underpinnings of extinction. Our aim is simplicity and generality. We will, for example, not assume *a priori* that the incident field is monochromatic or a plane wave, although these special cases are considered separately. When we do introduce approximations and special cases, our main condition for including a particular topic is broad applicability. Although all relations we derive are amenable to computations, we do not veer off into numerical analysis, as this is a separate large field. Rather, we aim to provide "actionable" formulas, which can be evaluated by a variety of methods. However, we give some references for the reader who wishes to implement some of the results numerically.

An important theoretical tool that we use throughout this chapter is the T-matrix of the target. One difficulty facing us here is a rather narrow view of the T-matrix, which is widespread in computational light scattering and adjacent fields. While the T-matrix has been known for a long time in physics (and is in fact a mathematical embodiment of linearity of the underlying partial differential equations), it has been introduced to the light scattering literature by Waterman in 1960-ies [8, 9]. In Waterman's formulation, the T-matrix is a matrix of coefficients, which couples the expansions of the scattered and incident fields in vector spherical harmonics. This understanding of the T-matrix became standard and is frequently confused with a more fundamental definition. While it is fruitless to argue which quantity should be called the T-matrix, it is important to recognize that linearity of Maxwell's equations guarantees that there exists a linear integral operator, which couples the induced polarization in the target to the incident field. Waterman's construct is a particular representation of this operator in the orthonormal basis of vector spherical harmonics. However, the operator itself exists regardless of any basis, and we refer to this integral operator as the T-matrix. We are free of course to use the basis of vector spherical harmonics to compute the T-matrix, but this is neither a requirement nor a definition. There exist other convenient bases that can be used in computations such as the basis of voxels (discrete dipoles).

Unlike many classical texts, we devote a substantial effort to cover extinction of partially-coherent waves (Sec. 5). This topic attracted significant recent attention but is not adequately covered in the textbooks. In the related sections, we follow the standard partial coherence theory developed by Wolf [10–12] and followers [13–15], but with some modifications. One such modification is that we express all measurable quantities as explicit time averages and do not introduce ensembles or ensemble averaging. Although, for ergodic processes, the two averaging approaches are equivalent, time-averaging is conceptually simpler and more physically relevant as measurements are performed on a particular physical system, not on an ensemble

of such systems, which may not even exist. Second, we define the coherence matrix as a correlation function of the real-valued physical field rather than an analytic signal. In doing so, we trade mathematical convenience of working with analytic signals for physical transparency. Besides, analytic signals of stationary stochastic processes is a dubious concept even from the mathematical point of view. We therefore make the choice to not introduce superfluous or non-existing entities for the sake of convenience. Finally, a very minor departure from the conventional approach is our use of the factor of 2π in the definition of cross spectral density. This is done for consistency with other conventionally used temporal Fourier transforms.

The final part of the chapter (Sec. 6) is devoted to the operational definition of extinction, that is, the definition of extinction cross section in terms of a measurement performed with a single flat, power-integrating detector, and to the associated paradoxes. We note that these topics continued to attract attention recently, with a defining contribution to the operational definition theory made in 2009 [16] and the classical extinction paradox revisited in 2011 [17].

We urge the reader to go through the opening section on notations (Sec. 2) before reading any of the subsequent chapters. The very last section (Sec. 7) contains auxiliary formulas and proofs, which can be of interest to some readers but are not essential to the main developments of this chapter.

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2 Notations

(a) Physical units

Gaussian system of units is used throughout. Accordingly, the dielectric permittivity and magnetic permeability of vacuum are equal to 1. For non-magnetic materials, which are exclusively considered in this chapter, the magnetic induction and magnetic field are identical. This is expressed as $\mathbf{B}(\mathbf{r}, t) = \mathbf{H}(\mathbf{r}, t)$. We use the notation $\mathbf{B}(\mathbf{r}, t)$ to represent the magnetic field.

(b) Vector notations

Three-dimensional vectors are denoted by bold font, which can be straight as in $\mathbf{E}$ or tilted as in $\boldsymbol{E}$. The difference is that the straight bold letters denote real-valued quantities while the tilted letters are reserved for vectors that might have complex components. Physical (that is, measurable) fields are all real-valued by definition and we use straight letters for them as in $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{S}$. This does not apply to various complex amplitudes, which are denoted using slanted bold fonts. In general, the chapter makes a consistent distinction between real and complex quantities. In some

instances we will denote real and imaginary parts of a quantity by superscripts (r) and (i) as for example in $\mathtt{W} = \mathtt{W}^{(\mathrm{r})} + \mathrm{i}\,\mathtt{W}^{(\mathrm{i})}$.

Cartesian components of vectors are denoted as other scalars by the Italic font as in E_i, $i = 1, 2, 3$ and sometimes as E_x, E_y and E_z if a Cartesian reference frame XYZ has been introduced. The radius-vector (the vector of position) is denoted by $\mathbf{r}$ and vectors of unit length indicating direction by $\mathbf{s}$, $\mathbf{n}$ or $\mathbf{u}$. The unit vectors along the axes of a Cartesian reference frame XYZ are denoted by $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$,

(c) Tensor notations

Tensors acting in the physical three-dimensional space are denoted by a straight, capital (sometimes, Greek), typewriter-style font as in $\mathtt{G}$ or $\mathtt{T}$ or Γ. Matrix elements of tensors, as all scalars, are denoted by an Italic font, as in G_{ij}. The unit tensor is denoted by $\mathtt{I}_3$. Overhead decorations are not used to distinguish tensors from scalars. The dot product of two vectors is denoted by a centered dot and the tensor product by an $\otimes$ symbol. For example, $\mathbf{f} \cdot \mathbf{g} = \sum_i f_i g_i$ and $\mathbf{f} \otimes \mathbf{g}$ is a tensor with the components $f_i g_j$. Note that the dot product of two vectors is different from the scalar product of two vector functions in a Hilbert space as defined in Paragraph (d) below. The dot product, unlike the scalar product, does not involve complex conjugation of vector elements.

Element-wise complex conjugation of a vector or a tensor is denoted by a star; if $\mathtt{B} = \mathtt{A}^*$, then $B_{ij} = A_{ij}^*$. Matrix transposition is denoted by a superscript T; if $\mathtt{B} = \mathtt{A}^{\mathrm{T}}$, then $B_{ij} = A_{ji}$. Hermitian conjugation is denoted by a dagger symbol; if $\mathtt{B} = \mathtt{A}^{\dagger}$, then $B_{ij} = A_{ji}^*$. Note that $\mathtt{A}^{\dagger} = (\mathtt{A}^{\mathrm{T}})^* = (\mathtt{A}^*)^{\mathrm{T}}$.

(d) Operator notations

We will sometimes work with linear operators whose integral kernels are tensors of the form $\mathtt{T}(\mathbf{r}, \mathbf{r}'; \cdot)$. Here $\cdot$ is a placeholder for an additional label on which the operator may depend; it can be frequency ω or time τ. The corresponding operator acts on the Hilbert space $\mathcal{H}(\mathbb{V})$ of square-integrable vector functions supported in the region $\mathbb{V}$, which is occupied by the scattering target. If two functions $\mathbf{f}(\mathbf{r})$ and $\mathbf{g}(\mathbf{r})$ are elements of $\mathcal{H}(\mathbb{V})$, the following expressions are equivalent:

$$\mathbf{f}(\mathbf{r}) = \int_{\mathbb{V}} \mathtt{T}(\mathbf{r}, \mathbf{r}'; \cdot)\mathbf{g}(\mathbf{r}')\mathrm{d}^3 r' \,, \tag{2.1a}$$

$$f_i(\mathbf{r}) = \sum_{j=1}^{3} \int_{\mathbb{V}} T_{ij}(\mathbf{r}, \mathbf{r}'; \cdot) g_j(\mathbf{r}')\mathrm{d}^3 r' \,, \tag{2.1b}$$

$$|\mathtt{f}\rangle = \mathtt{T}(\cdot)|\mathtt{g}\rangle \,. \tag{2.1c}$$

The scalar product of two vectors in $\mathcal{H}(\mathbb{V})$ is defined as

$$\langle g|f\rangle = \int_{\mathbb{V}} \mathbf{g}^*(\mathbf{r}) \cdot \mathbf{f}(\mathbf{r}) \mathrm{d}^3 r \ . \tag{2.2}$$

Note the complex conjugation, which is used in the definition of the scalar product, as above, but not in the definition of the dot product of two three-dimensional vectors, $\mathbf{f}(\mathbf{r}) \cdot \mathbf{g}(\mathbf{r}) = \sum_i f_i(\mathbf{r}) g_i(\mathbf{r})$.

To recount, the expression $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \cdot)$ is a 3×3 tensor that depends on all its arguments whereas $\mathsf{T}(\cdot)$ is an integral operator acting on $\mathcal{H}(\mathbb{V})$, which depends on one argument, whose label will appear in place of the centered dot.

Finally, we denote the unit operator in $\mathcal{H}(\mathbb{V})$ by $\mathsf{I}_{\mathbb{V}}$.

(e) Averaging

Time averaging is denoted by angle brackets as in $\langle ... \rangle$ and the average itself by an overhead bar. Thus, if $f(t)$ is a function of time, we write $\overline{f} = \langle f(t) \rangle$. The distinction here is important; overhead bar is not an operation of averaging and the quantity under the bar is not time-dependent. Rather, $\overline{f}$ denotes a special value of the function $f(t)$. For a stationary stochastic process $f(t)$, we define the time-average as

$$\overline{f} = \langle f(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} f(t) \mathrm{d}t \ . \tag{2.3}$$

(f) Fourier transforms

With the exception of the coherence matrix Γ and its temporal Fourier transform known as the cross-spectral density W, functions that are Fourier transforms of each other are denoted by the same symbol and distinguished only by the list of arguments. For temporal Fourier transforms, the convention used is

$$f(\omega) = \int_{-\infty}^{\infty} f(t) e^{\mathrm{i}\,\omega t} \mathrm{d}t \ , \quad f(t) = \int_{-\infty}^{\infty} f(\omega) e^{-\mathrm{i}\,\omega t} \frac{\mathrm{d}\omega}{2\pi} \ . \tag{2.4}$$

Variables of the dimensionality of time are denoted by t, τ, sometimes η and ζ (in nested time integrals). Frequency is denoted by ω.

In the case of the coherence matrix and the cross-spectral density, we follow the long-standing tradition and give the two quantities different names, even though they are temporal Fourier transforms of each other.

For spatial Fourier transforms, the convention used is

$$f(\mathbf{q}) = \int_{\mathbb{V}} f(\mathbf{r}) e^{-\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} \mathrm{d}^3 r \tag{2.5a}$$

for functions of position and

$$f(\mathbf{q}, \mathbf{q}') = \int_{\mathbb{V}} d^3r \int_{\mathbb{V}} d^3r' \, f(\mathbf{r}, \mathbf{r}') e^{i(\mathbf{q}' \cdot \mathbf{r}' - \mathbf{q} \cdot \mathbf{r})} \qquad (2.5b)$$

for kernels of integral operators. Integration in (2.5) is over the region occupied by the target, $\mathbb{V}$. We use the variables $\mathbf{q}$ and $\mathbf{k}$ as reciprocals to the radius-vector $\mathbf{r}$. Note that, unlike the temporal Fourier transforms (2.4), the spatial Fourier transforms used in this chapter are not generally invertible.

(g) Current and flux

We refer to the time-averaged Poynting vector $\overline{\mathbf{S}}(\mathbf{r})$ as the current of energy. Projection of this current onto a direction is the flux of energy through a surface that is perpendicular to that direction. For example, $\overline{\mathbf{S}}(x, y, z = L) \cdot \mathbf{e}_z$ is the flux of energy through the surface $z = L$. An integral of the form

$$\oint_{\partial\Omega} \overline{\mathbf{S}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, d^2r \qquad (2.6)$$

is the total outward flux of energy through the surface $\partial\Omega$, which encloses the three-dimensional region Ω. Here $\mathbf{n}(\mathbf{r})$ is the outward unit normal to $\partial\Omega$ at the point $\mathbf{r} \in \partial\Omega$.

(h) Special notations

The region occupied by the scatterer (target) is denoted by $\mathbb{V}$ and the volume of this region by $V[\mathbb{V}]$. Some other special notations used in the chapter include i for the imaginary unit (not to be confused with the indexes i and j, which are used to label Cartesian components of three-dimensional vectors), I_3 for the 3×3 unit tensor acting in the physical three-dimensional space, $I_{\mathbb{V}}$ for the unit operator acting in the Hilbert space $\mathcal{H}[\mathbb{V}]$ of square-integrable in $\mathbb{V}$ vector functions, $\Theta(x)$ for the unit step function, $\delta(x)$ for the Dirac delta-function, and δ_{ij} for the Kronecker delta-symbol. We denote Poynting vector by $\mathbf{S}(\mathbf{r}, t)$ (a vector) and the power spectrum by $S(\mathbf{r}, \omega)$ (a scalar). These quantities are not Fourier transforms of each other, and use the same symbol due to a long-standing tradition. The implied physical meaning should always be clear from the context. Speed of light in vacuum is denoted by c and the free-space wave number at the frequency ω by $k = \omega/c$.

3 The scattering problem

3.1 Model and assumptions

In a typical formulation of the scattering problem, a stationary electromagnetic field $\mathbf{E}_{\mathrm{inc}}(\mathbf{r}, t)$, $\mathbf{B}_{\mathrm{inc}}(\mathbf{r}, t)$ is incident onto a material object (the target) as illustrated schematically in Fig. 1. Interaction of the incident field with the target produces the scattered field $\mathbf{E}_{\mathrm{sca}}(\mathbf{r}, t)$, $\mathbf{B}_{\mathrm{sca}}(\mathbf{r}, t)$, and the total field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ (denoted without a subscript) is a superposition of the incident and scattered components, viz,

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_{\mathrm{inc}}(\mathbf{r}, t) + \mathbf{E}_{\mathrm{sca}}(\mathbf{r}, t) \,, \quad \mathbf{B}(\mathbf{r}, t) = \mathbf{B}_{\mathrm{inc}}(\mathbf{r}, t) + \mathbf{B}_{\mathrm{sca}}(\mathbf{r}, t) \,. \tag{3.1}$$

We assume that, at sufficiently high frequencies that are of interest in scattering experiments, the target is non-magnetic and completely characterized by the dielectric permittivity function $\epsilon(\mathbf{r}, \omega)$ or by the corresponding susceptibility function

$$\chi(\mathbf{r}, \omega) = \frac{\epsilon(\mathbf{r}, \omega) - 1}{4\pi} \,. \tag{3.2}$$

Note that the susceptibility (3.2) is identically zero in vacuum.

We will now discuss the physical model of this chapter and the involved assumptions in more detail.

First, the electromagnetic field consists of the electric component $\mathbf{E}$ and the magnetic component $\mathbf{B}$. As the vacuum and the target are both non-magnetic, we have, using the conventional notations, $\mathbf{B} = \mathbf{H}$ at any point in space and time. Although, under the assumptions made, there is no distinction between $\mathbf{B}$ and $\mathbf{H}$, we work with the physical field $\mathbf{B}$, which appears in the expression for the Lorentz force. Note however that $\mathbf{B}$ is customarily referred to as the magnetic induction and $\mathbf{H}$ as the magnetic field.

Second, the incident fields $\mathbf{E}_{\mathrm{inc}}$, $\mathbf{B}_{\mathrm{inc}}$ satisfy the free-space Maxwell's equations

$$\nabla \times \mathbf{B}_{\mathrm{inc}} = \frac{1}{c} \frac{\partial \mathbf{E}_{\mathrm{inc}}}{\partial t} \,, \quad \nabla \times \mathbf{E}_{\mathrm{inc}} = -\frac{1}{c} \frac{\partial \mathbf{B}_{\mathrm{inc}}}{\partial t} \,, \quad \nabla \cdot \mathbf{E}_{\mathrm{inc}} = \nabla \cdot \mathbf{B}_{\mathrm{inc}} = 0 \,. \tag{3.3}$$

These equations do not contain a source term. Therefore, applying physically-reasonable boundary conditions at infinity, one can conclude that (3.3) have only the trivial solution. For example, an infinite plane wave, although satisfies (3.3), is not a physical solution since it does not satisfy the conditions at infinity and, in reality, such infinite fields do not exist. To resolve the contradiction, we recall that the source-free equations (3.3) hold only in a finite region of space denoted here by $\mathbb{D}$. This region is sufficiently large to contain the target and the measurement devices, but there must necessarily exist non-zero source currents outside of $\mathbb{D}$. Importantly, we assume no back-action of the scattered field on the source current, so that the incident field can be viewed as given and known. If this assumption does not hold, the scattering problem becomes much less tractable. See [18] for additional considerations related to the distinction between the external (source) and induced currents.
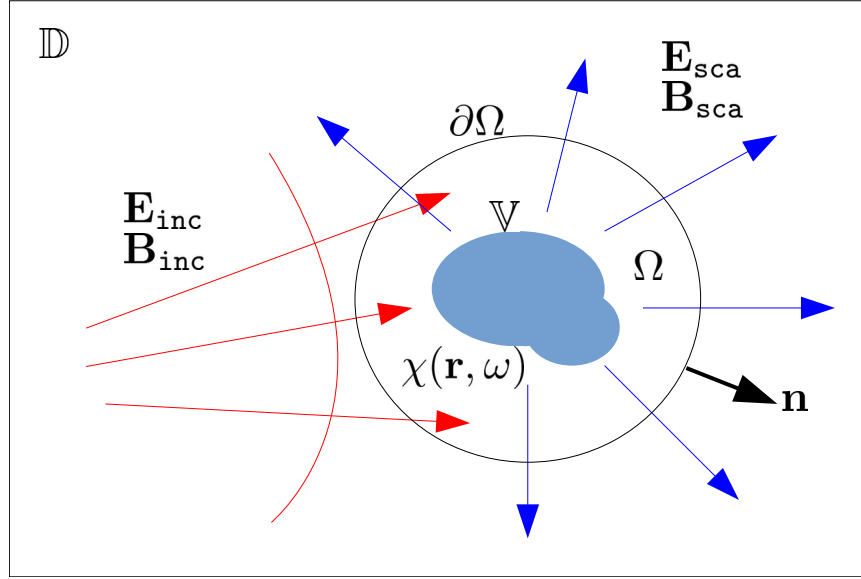
Fig. 1 Schematic illustration of the scattering problem. The incident field satisfies free-space Maxwell's equations in the region $\mathbb{D}$. The target occupies the region $\mathbb{V}$. Energy budgets are computed on the surface $\partial\Omega$, which is the boundary of the region Ω. It is assumed that Ω contains the target completely. Note the relation $\mathbb{V} \in \Omega \in \mathbb{D}$. The target is characterized by susceptibility $\chi(\mathbf{r}, \omega)$ in frequency domain or $\chi(\mathbf{r}, \tau)$ in time domain and has the spatial support in $\mathbb{V}$.

Third, we do not assume at the moment that the incident field is monochromatic or a plane wave, although we consider these special cases below. However, we require that it be stationary. If the target is passive (contains no external energy sources), this assumption implies that the scattered and the total fields are also stationary. By stationarity we mean that certain energy-related time averages exist and are independent of the averaging interval as long as it is sufficiently large [1]. For example, we require that the time average of the Poynting vector at a point $\mathbf{r} \in \mathbb{D}$,

$$\overline{\mathbf{S}}(\mathbf{r}) = \frac{c}{4\pi} \langle \mathbf{E}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t) \rangle = \frac{c}{4\pi} \frac{1}{T} \int\limits_{-T/2}^{T/2} \mathbf{E}(\mathbf{r}, t + \tau) \times \mathbf{B}(\mathbf{r}, t + \tau) \mathrm{d}\tau , \quad (3.4)$$

be a well-defined measurable quantity, which is independent of t as long as T is sufficiently large. For an additional explanation of the time averaging-related notations, see Sec. 2(d). Note also that the average in (3.4) can be expressed in terms of the field coherence matrix, which is introduced below in Sec. 5.2.

Fourth, we have introduced $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$ as functions of position and time, but the constitutive parameters $\epsilon(\mathbf{r}, \omega)$ and $\chi(\mathbf{r}, \omega)$ as functions of position and

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[1] This condition is weaker than strict mathematical stationarity of a stochastic process but sufficient in practice.

frequency. These notations can be reconciled as follows. The permittivity and susceptibility can be viewed as either functions of frequency or time, with the two representations related to each other by a Fourier transform. The time-domain representation of the susceptibility is often referred to as the influence function, and we denote it here by $\chi(\mathbf{r}, \tau)$. In general, we have

$$\mathbf{J}_{\text{ind}}(\mathbf{r}, t) = \frac{\partial}{\partial t}\mathbf{P}(\mathbf{r}, t) , \quad \mathbf{P}(\mathbf{r}, t) = \int_0^\infty \chi(\mathbf{r}, \tau)\mathbf{E}(\mathbf{r}, t - \tau)\mathrm{d}\tau . \qquad (3.5)$$

This functional form is appropriate for isotropic, non-magnetic media without spatial dispersion. The vector field $\mathbf{P}(\mathbf{r}, t)$ in (3.5) is the electric polarization of the target. The connection between the time-domain susceptibility (the influence function) and the frequency-domain susceptibility is established by the relation

$$\chi(\mathbf{r}, \omega) = \int_0^\infty \chi(\mathbf{r}, \tau)e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau . \qquad (3.6a)$$

Since integration in (3.6a) involves only positive τ, the frequency-domain susceptibility $\chi(\mathbf{r}, \omega)$ has no singularities in the upper half of the complex ω-plane. We can extend the integration in (3.6a) to the whole real axis by requiring that $\chi(\mathbf{r}, \tau) = 0$ for $\tau < 0$. With this additional condition, (3.6a) can be inverted as

$$\chi(\mathbf{r}, \tau) = \frac{1}{2\pi}\int_{-\infty}^\infty \chi(\mathbf{r}, \omega)e^{-\mathrm{i}\,\omega\tau}\mathrm{d}\omega = \frac{1}{\pi}\text{Re}\int_0^\infty \chi(\mathbf{r}, \omega)e^{-\mathrm{i}\,\omega\tau}\mathrm{d}\omega . \qquad (3.6b)$$

Note that $\chi(\mathbf{r}, \tau)$ is real whereas $\chi(\mathbf{r}, \omega)$ can be complex. The second equality in (3.6b) was obtained by accounting for the symmetry property $\chi(\mathbf{r}, -\omega) = \chi^*(\mathbf{r}, \omega)$, which follows from (3.6a) and $\text{Im}\chi(\mathbf{r}, \tau) = 0$. An example of an invertible transformation between the time-domain and frequency-domain susceptibility functions is given in Sec. 7.1.

Fifth, we assume that the target has a finite spatial support $\mathbb{V}$, which means that $\chi(\mathbf{r}, \cdot) = 0$ for $\mathbf{r} \notin \mathbb{V}$ and $\chi(\mathbf{r}, \cdot) \neq 0$ for $\mathbf{r} \in \mathbb{V}$. Here the centered dot is a place holder for either τ or ω. Note that $\mathbb{V}$ does not need to be connected and can consist of several disjoint regions. In many applications, $\chi(\mathbf{r}, \cdot)$ is spatially-uniform in $\mathbb{V}$. While we do not make this assumption *a priori*, it will be used in Sec. 5.9 on quasi-static approximation.

Finally, we have defined above two spatial regions, $\mathbb{D}$ and $\mathbb{V}$. The incident field satisfies the free-space Maxwell equations (3.3) in $\mathbb{D}$ and the sources of radiation are located outside of $\mathbb{D}$. The region $\mathbb{V} \in \mathbb{D}$ is occupied by the target. In what follows, we will also need to define another region of space, Ω, on the boundary of which we compute energy budgets. We require that Ω contain the target completely, be simply connected, convex, and that its boundary $\partial\Omega$ be smooth (except, perhaps, for a set of points of zero measure); otherwise Ω is arbitrary. If $\mathbb{V}$ satisfies the

convexity, smoothness and connectivity properties, we can, in principle, choose $\Omega = \mathbb{V}$. However, it is often convenient to make Ω a sphere of a large radius R. This may be inconsistent with $\Omega = \mathbb{V}$. In general, we require that $\mathbb{V} \in \Omega \in \mathbb{D}$. All three regions are illustrated schematically in Fig. 1.

The above physical model and assumptions set a convenient stage for theoretical investigation of the phenomena of electromagnetic extinction, scattering and absorption.

3.2 Absorption, scattering and extinction

Most textbooks characterize targets by absorption, scattering and extinction cross sections. However, cross sections can be defined unambiguously only if the incident field is a plane wave. Here we use a more general approach and define the absorbed, scattered and extinguished powers. In the case of an incident plane wave, these powers can be normalized by the incident energy flux to yield the conventional cross sections.

The only quantity that is defined in terms of a physical energy flux is the absorbed power, Q_{abs}. By *physical* we mean here the energy flux associated with the total electromagnetic field $\mathbf{E}$, $\mathbf{B}$, which is a superposition of the incident and scattered components according to (3.1). We have introduced above the region of space Ω, which contains the target and, possibly, some empty space (see Fig. 1). As the surface of this region $\partial\Omega$ is smooth almost everywhere, we can define the outward unit normal to $\partial\Omega$, $\mathbf{n}(\mathbf{r})$, at almost any point $\mathbf{r} \in \partial\Omega$. The absorbed power (energy absorbed by the target per unit time) is then given by the following integral:

$$Q_{\text{abs}} = - \oint_{\partial\Omega} \overline{\mathbf{S}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, \mathrm{d}^2 r \, , \qquad (3.7)$$

where the time-averaged Poynting vector $\overline{\mathbf{S}}(\mathbf{r})$ is defined in (3.4). Thus, the absorbed power is equal to the electromagnetic energy that enters Ω through its surface per unit time. Since the field is stationary, all this energy is absorbed by the target and converted into heat and there is no other channel of energy dissipation in the model.

The definition of scattered power Q_{sca} contains only the scattered field:

$$Q_{\text{sca}} = \oint_{\partial\Omega} \overline{\mathbf{S}}_{\text{sca}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, \mathrm{d}^2 r \, , \qquad (3.8)$$

where

$$\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r}) = \frac{c}{4\pi} \langle \mathbf{E}_{\text{sca}}(\mathbf{r}, t) \times \mathbf{B}_{\text{sca}}(\mathbf{r}, t) \rangle \qquad (3.9)$$

is a hypothetical Poynting vector associated with the scattered field alone. According to the intuitive understanding, the scattered power is the energy leaving the region Ω

through its surface per unit time and computed under the assumption that only the scattered field is present on $\partial\Omega$. Although this assumption never holds precisely, we can seek a measurement scheme in which it holds approximately or otherwise try to access Q_{sca} in an indirect measurement.

Finally, the extinguished power Q_{ext} is, by definition, the sum of the absorbed and scattered powers:

$$Q_{\text{ext}} = Q_{\text{abs}} + Q_{\text{sca}} \ . \tag{3.10}$$

Using expressions (3.7), (3.8), the decomposition of the total field (3.1), and accounting for the fact that the incident field has zero total energy flux through $\partial\Omega$ (otherwise, empty space would absorb or generate energy), we obtain

$$Q_{\text{ext}} = - \oint_{\partial\Omega} \overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, \mathrm{d}^2 r \ , \tag{3.11}$$

where

$$\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) = \frac{c}{4\pi} \langle \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \times \mathbf{B}_{\text{sca}}(\mathbf{r}, t) + \mathbf{E}_{\text{sca}}(\mathbf{r}, t) \times \mathbf{B}_{\text{inc}}(\mathbf{r}, t) \rangle \ . \tag{3.12}$$

Equation (3.12) indicates that extinction is an interference effect. The extinguished power is the time-averaged total inward flux through $\partial\Omega$ of the peculiar (and, notably, not directly measurable) energy current $\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r})$ given by a quadratic combination of the incident and the scattered fields.

We thus see that, if it was possible to spatially separate the incident and scattered fields everywhere on $\partial\Omega$, extinction would be zero, which implies zero absorption and scattering (as both powers are defined as non-negative quantities). However, extinction by passive particles is almost never zero [2]. This means that the incident and scattered fields always overlap somewhere on $\partial\Omega$.

3.3 Local (volume) representation

Maxwell's equations for the total field $\mathbf{E}$, $\mathbf{B}$ are of the following form in $\mathbb{D}$:

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{J}_{\text{ind}} \ , \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \ ; \quad \mathbf{r} \in \mathbb{D} \ . \tag{3.13}$$

Here $\mathbf{J}_{\text{ind}}$ is the electric current induced in the target according to the constitutive relation (3.5). Let us dot-multiply the first equation in (3.13) by $\mathbf{B}$, the second

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[2] Here we do not discuss so-called electromagnetic cloaks whose extinction is, at least theoretically, zero [19]. For a given illumination (say, with a tightly-collimated beam), vanishing extinction is easy to achieve using several high-quality mirrors. The mirrors in this case must be included in the definition of the target. It is much harder if at all possible to achieve vanishing extinction under all possible external illuminations.

equation by $\mathbf{E}$, and subtract the latter from the former. Accounting for the vector identity $\nabla(\mathbf{E} \times \mathbf{B}) = \mathbf{B} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{B}$, we arrive at the Poynting theorem

$$\frac{c}{4\pi}\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \frac{1}{8\pi}\frac{\partial}{\partial t}(\mathbf{E} \cdot \mathbf{E} + \mathbf{B} \cdot \mathbf{B}) + \mathbf{J}_{\text{ind}} \cdot \mathbf{E} = 0 \ . \tag{3.14}$$

At the next step, we time-average (3.14). Since all fields are assumed to be stationary, averaging of the time derivative yields zero. Using the definition of $\overline{\mathbf{S}}(\mathbf{r})$ (3.4), we obtain the relation

$$\nabla \cdot \overline{\mathbf{S}}(\mathbf{r}) + \langle \mathbf{J}_{\text{ind}}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \rangle = 0 \ . \tag{3.15}$$

We then integrate (3.15) over the three-dimensional region Ω, use Gauss theorem for the divergence term and, comparing to (3.7), find the volume-integral expression for the absorbed power:

$$Q_{\text{abs}} = \int_{\mathbb{V}} \langle \mathbf{J}_{\text{ind}}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \rangle \, \mathrm{d}^3 r \ . \tag{3.16}$$

Note that integration in (3.16) is over $\mathbb{V} \in \Omega$ rather than over Ω because $\mathbf{J}_{\text{ind}}(\mathbf{r}, t) = 0$ for $\mathbf{r} \notin \mathbb{V}$.

From elementary mechanical considerations, we know that $\langle \mathbf{J}_{\text{ind}}(\mathbf{r}, t) \cdot \mathbf{E}(\mathbf{r}, t) \rangle$ is the time-averaged volume density of the work done by the electric field $\mathbf{E}(\mathbf{r}, t)$ on the charges of the medium whose motion makes up the induced current $\mathbf{J}_{\text{ind}}(\mathbf{r}, t)$. Since we consider a stationary process, the average electromagnetic energy density does not change anywhere in space, and the only way to reconcile this with the first law of thermodynamics is to conclude that the mechanical work of the electric field is entirely converted into heat. For this reason, the integrand in (3.16) is known as Joule heat.

Equation (3.16) expresses the law of energy conservation and is well known. To derive a similar volume integral representation for the extinguished power, we notice that the scattered field, not unexpectedly, satisfies the same equations as the total field, that is, (3.13). Indeed, substituting the decomposition (3.1) into (3.13) and accounting for the fact that the incident field satisfies the homogeneous equations (3.3) in $\mathbb{D}$, we find that

$$\nabla \times \mathbf{B}_{\text{sca}} = \frac{1}{c}\frac{\partial \mathbf{E}_{\text{sca}}}{\partial t} + \frac{4\pi}{c}\mathbf{J}_{\text{ind}} \ , \quad \nabla \times \mathbf{E}_{\text{sca}} = -\frac{1}{c}\frac{\partial \mathbf{B}_{\text{sca}}}{\partial t} \ ; \quad \mathbf{r} \in \mathbb{D} \ . \tag{3.17}$$

Of course, we should keep in mind that the induced current $\mathbf{J}_{\text{ind}}$ in (3.17) is determined by the constitutive relations (3.5), which involve the total field, not just its scattered component. For this reason, (3.17) is not closed. However, following the same steps as above, and using (3.8), we arrive at

$$Q_{\text{sca}} = -\int_{\mathbb{V}} \langle \mathbf{J}_{\text{ind}}(\mathbf{r}, t) \cdot \mathbf{E}_{\text{sca}}(\mathbf{r}, t) \rangle \, \mathrm{d}^3 r \ . \tag{3.18}$$

Adding (3.16) with (3.18) together, we find that

$$Q_{\text{ext}} = \int\limits_{\mathbb{V}} \langle \mathbf{J}_{\text{ind}}(\mathbf{r}, t) \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \rangle \, \mathrm{d}^3 r \; . \qquad (3.19)$$

It follows that the extinguished power is the time-averaged total work done by the incident field on the moving charges of the target.

Thus, we have shown that the absorbed, scattered and extinguished powers can be represented either as total energy fluxes through the surface $\partial\Omega$, which can be arbitrarily far away from the target (equations (3.7), (3.8), (3.11)), or as volume integrals over the target support $\mathbb{V}$ (equations (3.16), (3.18), (3.19)). We refer to these representations as to *global* and *local*. Note that all the expressions derived above hold for constitutive relations of a more general form than (3.5). In particular, they hold for magnetic materials. In the latter case, the induced current should include an additional term and be written as $\mathbf{J}_{\text{ind}} = \partial\mathbf{P}/\partial t + c\nabla \times \mathbf{M}$, where $\mathbf{M}$ is the vector of magnetization. A question may arise regarding the *differential density* of heat produced in magnetic targets [20–24]. However, there is no ambiguity regarding the *total* quantities that are defined in (3.16), (3.18) and (3.19). It can be concluded that the derivations of this section are not affected by accounting for magnetization.

4 Monochromatic fields

4.1 Time averaging for monochromatic fields

Any real-valued, monochromatic field $\mathbf{F}(\mathbf{r}, t)$ oscillating at the frequency ω can be written in the form

$$\mathbf{F}(\mathbf{r}, t) = \text{Re}[\boldsymbol{F}(\mathbf{r})e^{-\mathrm{i}\,\omega t}] = \frac{1}{2}[\boldsymbol{F}(\mathbf{r})e^{-\mathrm{i}\,\omega t} + \boldsymbol{F}^*(\mathbf{r})e^{\mathrm{i}\,\omega t}] \; , \qquad (4.1)$$

where the time-dependent factor $e^{-\mathrm{i}\,\omega t}$ is known as the phasor and $\boldsymbol{F}(\mathbf{r})$ can be complex. The function in the square brackets is known as the analytic signal. From (4.1), it is obvious that the analytic signal defines uniquely the physical field. Converse is not always true. However, for monochromatic fields and for fields with a discrete spectrum, which are considered in Sec. 4.8 below, there is no such difficulty. Indeed, (4.1) is simply the most general from of a monochromatic field. We can substitute this ansatz into the Maxwell's equations and solve for $\boldsymbol{F}(\mathbf{r})$, at least in principle, assuming the target geometry and properties at the working frequency are known. The physical fields are then obtained by taking the real parts. We will therefore represent all monochromatic fields in the form (4.1), i.e.,

$$\mathbf{E}_*(\mathbf{r}, t) = \text{Re}[\boldsymbol{E}_*(\mathbf{r})e^{-\mathrm{i}\,\omega t}] \; , \qquad \mathbf{B}_*(\mathbf{r}, t) = \text{Re}[\boldsymbol{B}_*(\mathbf{r})e^{-\mathrm{i}\,\omega t}] \; ; \qquad (4.2\text{a})$$

$$\mathbf{P}(\mathbf{r}, t) = \text{Re}[\boldsymbol{P}(\mathbf{r})e^{-\mathrm{i}\,\omega t}] \; , \qquad \mathbf{J}_{\text{ind}}(\mathbf{r}, t) = \text{Re}[\boldsymbol{J}_{\text{ind}}(\mathbf{r})e^{-\mathrm{i}\,\omega t}] \; . \qquad (4.2\text{b})$$

Here the star in the subscript is a placeholder for either "inc", "sca", "ext", or for no subscript at all in the case of the total field. Using the above formalism, time averaging can be performed easily, as all the quadratic combinations of interest can be decomposed into time-independent and fast-oscillating components, and the latter average to zero.

We will now show how the time averaging can be performed in the global and local representations of the absorbed, scattered and extinguished powers.

4.1.1 Global representation

For the global representation, we need to compute the time averages of energy currents $\overline{\mathbf{S}}$, $\overline{\mathbf{S}}_{\mathrm{sca}}$ and $\overline{\mathbf{S}}_{\mathrm{ext}}$, which are defined in equations (3.4), (3.9) and (3.12). Using the representation (4.2) for the fields in the above formulas, and keeping only the time-independent terms, we obtain

$$\overline{\mathbf{S}}(\mathbf{r}) = \frac{c}{8\pi}\mathrm{Re}[\boldsymbol{E}(\mathbf{r}) \times \boldsymbol{B}^*(\mathbf{r})] \,, \tag{4.3a}$$

$$\overline{\mathbf{S}}_{\mathrm{sca}}(\mathbf{r}) = \frac{c}{8\pi}\mathrm{Re}[\boldsymbol{E}_{\mathrm{sca}}(\mathbf{r}) \times \boldsymbol{B}^*_{\mathrm{sca}}(\mathbf{r})] \,, \tag{4.3b}$$

$$\overline{\mathbf{S}}_{\mathrm{ext}}(\mathbf{r}) = \frac{c}{8\pi}\mathrm{Re}[\boldsymbol{E}_{\mathrm{inc}}(\mathbf{r}) \times \boldsymbol{B}^*_{\mathrm{sca}}(\mathbf{r}) + \boldsymbol{E}_{\mathrm{sca}}(\mathbf{r}) \times \boldsymbol{B}^*_{\mathrm{inc}}(\mathbf{r})] \,. \tag{4.3c}$$

Substituting these results into (3.7), (3.8) and (3.11), we can express the absorbed, scattered and extinguished powers in terms of surface integrals. We will find expression (4.3b) particularly useful in Sec. 4.4.1 below. However, most of the theoretical developments of this chapter are based on the local representation, which we consider next.

4.1.2 Local representation

To perform time averaging in the local representations, we use the first equation in (3.5) to write a relation between the complex amplitudes $\boldsymbol{J}_{\mathrm{ind}}(\mathbf{r})$ and $\boldsymbol{P}(\mathbf{r})$, namely,

$$\boldsymbol{J}_{\mathrm{ind}}(\mathbf{r}) = -\mathrm{i}\,\omega\boldsymbol{P}(\mathbf{r}) \,. \tag{4.4}$$

We substitute this result into (3.16), (3.18) and (3.19), use (4.2) for the fields, keep only time-independent terms, and arrive at

$$Q_{\mathrm{abs}} = \frac{\omega}{2}\mathrm{Im}\int_{\mathbb{V}} \boldsymbol{P}(\mathbf{r}) \cdot \boldsymbol{E}^*(\mathbf{r})\,\mathrm{d}^3r \,, \tag{4.5a}$$

$$Q_{\mathrm{sca}} = -\frac{\omega}{2}\mathrm{Im}\int_{\mathbb{V}} \boldsymbol{P}(\mathbf{r}) \cdot \boldsymbol{E}^*_{\mathrm{sca}}(\mathbf{r})\,\mathrm{d}^3r \,, \tag{4.5b}$$

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} \boldsymbol{P}(\mathbf{r}) \cdot \boldsymbol{E}_{\text{inc}}^*(\mathbf{r}) \, d^3 r \; . \tag{4.5c}$$

These expressions can be further developed by noting that

$$\boldsymbol{P}(\mathbf{r}) = \chi(\mathbf{r}, \omega) \boldsymbol{E}(\mathbf{r}) = \frac{\epsilon(\mathbf{r}, \omega) - 1}{4\pi} \boldsymbol{E}(\mathbf{r}) \; . \tag{4.6}$$

Using (4.6) in (4.5), we obtain

$$Q_{\text{abs}} = \frac{\omega}{8\pi} \int_{\mathbb{V}} \text{Im}[\epsilon(\mathbf{r}, \omega)] \, |\boldsymbol{E}(\mathbf{r})|^2 \, d^3 r \; , \tag{4.7a}$$

$$Q_{\text{sca}} = -\frac{\omega}{8\pi} \text{Im} \int_{\mathbb{V}} [\epsilon(\mathbf{r}, \omega) - 1] \, \boldsymbol{E}(\mathbf{r}) \cdot \boldsymbol{E}_{\text{sca}}^*(\mathbf{r}) \, d^3 r \; , \tag{4.7b}$$

$$Q_{\text{ext}} = \frac{\omega}{8\pi} \text{Im} \int_{\mathbb{V}} [\epsilon(\mathbf{r}, \omega) - 1] \, \boldsymbol{E}(\mathbf{r}) \cdot \boldsymbol{E}_{\text{inc}}^*(\mathbf{r}) \, d^3 r \; . \tag{4.7c}$$

Thus, the extinguished power for monochromatic fields is given by the imaginary part of the projection of the target susceptibility onto the interference pattern formed by the total and the incident fields. For a single plane wave, expression (4.7c) is equivalent to the optical theorem; see, for example [25] for discrete or [26] for continuous targets. For more general incident fields, this expression was derived in [27] where an additional term accounting for magnetization was also included. We consider the optical theorem in much more detail in Sec. 4.4 below.

4.2 Green's tensor and integral equations

There exists a simple mathematical technique to relate linearly the scattered field everywhere in space to the electric polarization field inside the target. We start from equations (3.17), which take the following form in the monochromatic case:

$$\nabla \times \boldsymbol{B}_{\text{sca}} = -i k \boldsymbol{E}_{\text{sca}} - 4\pi i k \boldsymbol{P} \; , \quad \nabla \times \boldsymbol{E}_{\text{sca}} = i k \boldsymbol{B}_{\text{sca}} \; ; \tag{4.8}$$
$$k = \omega/c \; , \quad \mathbf{r} \in \mathbb{D} \; .$$

Here k is the free space wave number at the working frequency. Note that equations (4.8) are written for the complex amplitudes. We have evaluated time derivatives of the phasor $e^{-i\omega t}$ and accounted for the relation (4.4). We say that equations (4.8) are written in frequency domain. Now it can be seen that the electric polarization $\boldsymbol{P}$ serves as the source for the scattered field. Physically, this makes sense: the electric oscillations induced in the target produce secondary waves. It is true that $\boldsymbol{P}$ is determined by the total field according to (4.6) rather than by the scattered field alone. For this reason, equations (4.8) are not closed, as was already mentioned. However,

we can formally invert (4.8) by viewing P as a free term. From the linearity of (4.8) and the uniqueness theorem for Maxwell's equations solution (when supplemented by the radiation-type boundary conditions at infinity), we conclude that the scattered field can be written as

$$E_{\mathrm{sca}}(\mathbf{r}) = \int_{\mathbb{V}} \mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega) P(\mathbf{r}') \mathrm{d}^3 r' \,, \quad B_{\mathrm{sca}}(\mathbf{r}) = -\mathrm{i} \, \nabla \times E_{\mathrm{sca}}(\mathbf{r}) \,, \quad \mathbf{r} \in \mathbb{D} \,. \quad (4.9)$$

Here $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega)$ is the free-space, frequency-domain Green's tensor. Its specific form can be found by substitution of (4.9) into (4.8); see, e.g., [28] for a derivation. It is also related by the temporal Fourier transform of the time-domain retarded Green's function, which is described in Sec. 5.3 below. The well-known expression for $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega)$ is

$$\mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega) = -\frac{4\pi}{3} \mathsf{I}_3 \delta(\mathbf{r} - \mathbf{r}') + \mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega) \,, \quad (4.10a)$$

where

$$\begin{aligned}
\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega) = \Bigg[& \left(\frac{k^2}{|\mathbf{r} - \mathbf{r}'|} + \frac{\mathrm{i}\, k}{|\mathbf{r} - \mathbf{r}'|^2} - \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \right) \mathsf{I}_3 \\
& + \left(-\frac{k^2}{|\mathbf{r} - \mathbf{r}'|} - \frac{3\mathrm{i}\, k}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{3}{|\mathbf{r} - \mathbf{r}'|^3} \right) \frac{(\mathbf{r} - \mathbf{r}') \otimes (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \Bigg] e^{\mathrm{i}\, k |\mathbf{r} - \mathbf{r}'|} \,. \quad (4.10b)
\end{aligned}$$

Here I_3 denotes the 3×3 identity matrix, the symbol $\otimes$ denotes tensor product of two vectors, and G_{R} is the "regular" part of the Green's tensor, as is indicated by the subscript. Regularity is understood as the property

$$\lim_{a \to 0} \int_{|\mathbf{r}'| < a} \mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r} + \mathbf{r}'; \omega) \, \mathrm{d}^3 r' = 0 \,. \quad (4.11)$$

We thus keep in mind that $\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega)$ still has a singularity when $\mathbf{r}' \to \mathbf{r}$; hence, we wrote "regular" in quotes. However, according to (4.11), the singularity integrates to zero over a small ball (but not necessarily over a small region of more general shape). A more mathematically-rigorous derivation of the integral relation (4.9) and and of the Green's tensor singular part is given in [29].

The free-space Green's tensor possesses a number of symmetries some of which are inherited from the *full* Green's tensor [3] and other are not. All these properties

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[3] The free-space Green's tensor gives the field produced by a point-like dipole source of unit magnitude in free space. The full Green's tensor gives the same but in the presence of the target. In the literature, the free-space and the full Green's tensors are often denoted by G_0 and G (we omit the argument ω in this footnote). The two quantities satisfy the formal relation $\mathsf{G} = \mathsf{G}_0 + \mathsf{G}_0 \mathsf{V} \mathsf{G} = \mathsf{G}_0 + \mathsf{G} \mathsf{V} \mathsf{G}_0$, which must be understood in the operator sense, where V is the interaction operator. The relation to the T-matrix is $\mathsf{V}\mathsf{G} = \mathsf{T}\mathsf{G}_0$ or $\mathsf{G}\mathsf{V} = \mathsf{G}_0\mathsf{T}$. The kernel of V depends on whether we include the singular part in the definition of G_0. If G_0 is the regular part of the free-space Green's tensor as given in (4.10b), then $\mathsf{V}(\mathbf{r}, \mathbf{r}'; \omega)$ is defined in Eq. (4.79) below. In this chapter, we do not use the

are also obeyed by the regular part of the free-space Green's tensor, G_R. First, G is translationally-invariant, which means that

$$G(\mathbf{r}, \mathbf{r}'; \omega) = G(\mathbf{r} + \mathbf{a}, \mathbf{r}' + \mathbf{a}; \omega) \qquad (4.12a)$$

for any constant vector $\mathbf{a}$. Thus, $G(\mathbf{r}, \mathbf{r}'; \omega)$ depends only on the shift $\mathbf{r} - \mathbf{r}'$. This property is a consequence of the translational invariance of free-space and is lost for the full Green's tensor. Second, changing the sign of the frequency results in element-wise complex conjugation of G, which is expressed as

$$G(\mathbf{r}, \mathbf{r}'; -\omega) = G^*(\mathbf{r}, \mathbf{r}'; \omega) \ . \qquad (4.12b)$$

This property is, of course, general and is retained by the full Green's tensor. Keep in mind that the asterisk in (4.12b) is an element-wise operation of complex conjugation without transposition. Third, G satisfies the relation

$$G(\mathbf{r}, \mathbf{r}'; \omega) = G^T(\mathbf{r}', \mathbf{r}; \omega) \ . \qquad (4.12c)$$

where the superscript T denotes matrix transposition. This property holds for the full Green's tensor and the T-matrix as well and is a mathematical expression of the principle of electromagnetic reciprocity. This principle is applicable provided that the constitutive relations for the target are reciprocal, which is, typically, the case in scattering experiments. In particular, the constitutive relations considered in the chapter are reciprocal. The property (4.12c) implies that any integral operator with the kernel $G(\mathbf{r}, \mathbf{r}'; \omega)$ is symmetric but not Hermitian. Correspondingly, the eigenvalues of such operators are complex, with imaginary parts describing radiative relaxation [26, 30]. The corresponding eigenvectors form a complete basis under typical circumstances, although a complex symmetric operator can be, in principle, defective. The eigenvectors are however not orthonormal in the usual sense. Instead, a slightly different orthogonality relation (not involving complex conjugation) holds.

The free-space Green's tensor possesses some symmetries in addition to (4.12c), viz,

$$G(\mathbf{r}, \mathbf{r}'; \omega) = G(\mathbf{r}', \mathbf{r}; \omega) = G^T(\mathbf{r}, \mathbf{r}'; \omega) \ . \qquad (4.12d)$$

The properties (4.12a) and (4.12d) are not inherited by the full Green's tensor or the T-matrix. However, they can be useful in numerical analysis. In particular, the translational invariance (4.12a) implies that a discretization of $G(\mathbf{r}, \mathbf{r}'; \omega)$ on a rectangular grid is a block-Toeplitz matrix and the property (4.12d) implies that this matrix is, in addition, symmetric [31].

For later reference, we will also write G_R in the following form:

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full Green's tensor and work instead with the T-matrix and the free-space Green's tensor. Therefore, we use the symbol G without the subscript "0" for the latter. This is done, in particular, because the notations G_0, G_1 and G_2 are used in the expansion of the algebraic part of the free-space Green's tensor in powers of $k = \omega/c$ in frequency domain or $\partial/\partial\tau$ in time domain, see equations (4.13) and (5.12) below. See Sec. 4.7 for formal algebraic relations between T, G and V.

$$\mathsf{G}_\mathrm{R}(\mathbf{r}, \mathbf{r}'; \omega) = \left[\mathsf{G}_0(\mathbf{r}, \mathbf{r}') + \mathrm{i}\, k\, \mathsf{G}_1(\mathbf{r}, \mathbf{r}') - k^2\, \mathsf{G}_2(\mathbf{r}, \mathbf{r}') \right] e^{\mathrm{i}\, k |\mathbf{r} - \mathbf{r}'|} \,, \qquad (4.13\mathrm{a})$$

where

$$\mathsf{G}_0(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \left(-\mathsf{I}_3 \ + \ 3 \frac{(\mathbf{r} - \mathbf{r}') \otimes (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \right) , \qquad (4.13\mathrm{b})$$

$$\mathsf{G}_1(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|^2} \left(\ \mathsf{I}_3 \ - \ 3 \frac{(\mathbf{r} - \mathbf{r}') \otimes (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \right) , \qquad (4.13\mathrm{c})$$

$$\mathsf{G}_2(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|} \left(-\mathsf{I}_3 \ + \ \frac{(\mathbf{r} - \mathbf{r}') \otimes (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^2} \right) . \qquad (4.13\mathrm{d})$$

Note that the tensors G_0, G_1 and G_2 are independent of ω (or $k = \omega/c$). The form (4.13) is convenient when used in time-frequency Fourier transforms.

From the general result (4.9), we can derive the integral equations for the electric polarization field in a few simple steps. First, we recall that $\boldsymbol{E}_\mathrm{sca}(\mathbf{r}) = \boldsymbol{E}(\mathbf{r}) - \boldsymbol{E}_\mathrm{inc}(\mathbf{r})$ and re-write the first equation in (4.9) as

$$\boldsymbol{E}(\mathbf{r}) = \boldsymbol{E}_\mathrm{inc}(\mathbf{r}) + \int_\mathbb{V} \mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{P}(\mathbf{r}') \mathrm{d}^3 r' \,; \quad \mathbf{r} \in \mathbb{D} \,. \qquad (4.14)$$

We then use the decomposition (4.10a) to account for the singularity of the Green's tensor. This yields

$$\boldsymbol{E}(\mathbf{r}) + \frac{4\pi}{3} \boldsymbol{P}(\mathbf{r}) = \boldsymbol{E}_\mathrm{inc}(\mathbf{r}) + \int_\mathbb{V} \mathsf{G}_\mathrm{R}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{P}(\mathbf{r}') \mathrm{d}^3 r' \,; \quad \mathbf{r} \in \mathbb{D} \,. \qquad (4.15)$$

Equations (4.14) and (4.15) are valid everywhere in $\mathbb{D}$, including the target interior and the free space. However, if we limit consideration to $\mathbf{r} \in \mathbb{V}$, where $\epsilon(\mathbf{r}) \neq 1$, then equation (4.6) can be inverted and we can express $\boldsymbol{E}$ in terms of $\boldsymbol{P}$ as $\boldsymbol{E} = [4\pi/(\epsilon - 1)]\boldsymbol{P}$. Substituting this result into the left-hand side of (4.15), we obtain

$$\frac{4\pi}{3} \frac{\epsilon(\mathbf{r}, \omega) + 2}{\epsilon(\mathbf{r}, \omega) - 1} \boldsymbol{P}(\mathbf{r}) = \boldsymbol{E}_\mathrm{inc}(\mathbf{r}) + \int_\mathbb{V} \mathsf{G}_\mathrm{R}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{P}(\mathbf{r}') \mathrm{d}^3 r' \,; \quad \mathbf{r} \in \mathbb{V} \,. \qquad (4.16)$$

Equation (4.16) is valid only in $\mathbb{V}$. But this is sufficient since $\mathbb{V}$ is the support of $\boldsymbol{P}(\mathbf{r})$ and (4.16) does not contain any other unknown functions. Now, assuming that $\epsilon(\mathbf{r}, \omega) \neq -2$ [4], we arrive at the final form of the integral equation:

$$\boldsymbol{P}(\mathbf{r}) = \kappa(\mathbf{r}, \omega) \left[\boldsymbol{E}_\mathrm{inc}(\mathbf{r}) + \int_\mathbb{V} \mathsf{G}_\mathrm{R}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{P}(\mathbf{r}') \mathrm{d}^3 r' \right] \,; \quad \mathbf{r} \in \mathbb{V} \,, \qquad (4.17)$$

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[4] There are no materials in nature for which $\epsilon(\omega)$ is equal to -2 *exactly* at any frequency. For metals, the real part of $\epsilon(\omega)$ can be equal to -2, but there is also a non-zero imaginary part. Otherwise, we would obtain many inconsistencies such as, for example, infinite polarizability of a small sphere.

where

$$\kappa(\mathbf{r}, \omega) = \frac{3}{4\pi} \frac{\epsilon(\mathbf{r}, \omega) - 1}{\epsilon(\mathbf{r}, \omega) + 2} \qquad (4.18)$$

is the *coupling function*.

It may seem that the transition from (4.15) to (4.16) requires that $|\epsilon(\mathbf{r}, \omega) - 1|$ be bounded from below in $\mathbb{V}$. This is actually not true. Derivation of equation (4.17) does not require the above condition to hold. In particular, we can include in $\mathbb{V}$ a region of empty space. The necessary and sufficient condition on $\mathbb{V}$ for (4.17) to hold is that $\mathbb{V}$ contain the target completely. Although including empty space in $\mathbb{V}$ seems to be superfluous, doing so can be useful in some numerical implementations of electromagnetic solvers. A case in point is the numerical technique known as the discrete-dipole approximation (DDA) [32–35] wherein a computationally-efficient matrix-vector multiplication is achieved by using fast Fourier transforms. For this to work, the target must be discretized on a rectangular box, which is not always commensurate with the target shape. In such cases, the box must include some empty space.

Assuming $\boldsymbol{E}_{\text{inc}}(\mathbf{r})$ is known, equation (4.17) defines $\boldsymbol{P}(\mathbf{r})$ uniquely in the sense that, in principle, it can be solved for $\boldsymbol{P}(\mathbf{r})$, i.e., numerically, and the solution is unique. Once $\boldsymbol{P}(\mathbf{r})$ is found, we can compute the extinguished power by applying (4.5c). The absorbed power can be computed from (4.5a) by expressing $\boldsymbol{E}(\mathbf{r})$ in terms of $\boldsymbol{P}(\mathbf{r})$. The resulting expressions are

$$Q_{\text{abs}} = 2\pi\omega \int_{\mathbb{V}} \frac{\text{Im}[\epsilon(\mathbf{r}, \omega)]}{|\epsilon(\mathbf{r}, \omega) - 1|^2} \, |\boldsymbol{P}(\mathbf{r})|^2 \, \mathrm{d}^3 r \,, \qquad (4.19a)$$

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} \boldsymbol{P}(\mathbf{r}) \cdot \boldsymbol{E}^*_{\text{inc}}(\mathbf{r}) \mathrm{d}^3 r \,. \qquad (4.19b)$$

Here we have repeated equation (4.5c) for completeness. The scattered power can be computed as the difference between Q_{ext} and Q_{abs}.

Thus, solving to the integral equation (4.17) allows one to compute all three powers considered above as quadratures. However, excluding the case of ellipsoidal targets, analytical solutions to (4.17) are not known. In most cases, (4.17) must be solved numerically. The approach of discrete dipole approximation (DDA) is to discretize (4.17) on some grid and convert it to a set of algebraic equations, which can be solved by the methods of linear algebra.

4.2.1 First Born approximation

Assuming that the second term in the right-hand side of (4.17) is in some sense small, we can develop a perturbation expansion for $\boldsymbol{P}$. Considering the latter equation as an iterative definition of $\boldsymbol{P}(\mathbf{r})$ in terms of itself, we can write

$$P(\mathbf{r}) = \kappa(\mathbf{r}, \omega) E_{\text{inc}}(\mathbf{r}) + \int_{\mathbb{V}} \kappa(\mathbf{r}, \omega) \mathsf{G}_{\text{R}}(\mathbf{r}, \mathbf{r}'; \omega) \kappa(\mathbf{r}', \omega) E_{\text{inc}}(\mathbf{r}') \mathrm{d}^3 r' + \dots$$

(4.20)

Keeping only the first term in the right-hand side of (4.20) yields an especially simple approximation for Q_{abs} and Q_{ext}. Indeed, substituting $P = \kappa E_{\text{inc}}$ into (4.19) yields

$$Q_{\text{abs}} = Q_{\text{ext}} \approx \frac{\omega}{2} \int_{\mathbb{V}} \text{Im}[\kappa(\mathbf{r}, \omega)] \, |E_{\text{inc}}(\mathbf{r})|^2 \mathrm{d}^3 r \, .$$

(4.21)

This approximation is valid for small or low-contrast targets when scattering is negligible; it is conventionally referred to as the first Born approximation. Note that (4.21) is different from the naive first-order approximation in which we replace P in (4.5c) with $[(\epsilon - 1)/4\pi]\mathbf{E}_{\text{inc}}$. By comparing to (4.21), it can be seen that the two approximations differ by the factor $\text{Im}[3(\epsilon-1)/(\epsilon+2)]/\text{Im}(\epsilon-1)$ inside the integral. The difference is due to accounting in (4.21) for the singularity of the Green's tensor, and is known in various contexts as the Lorentz local field correction. The correction is physically meaningful and allows one to obtain more accurate results. Thus, for a small homogeneous sphere of radius a, (4.21) yields the exact result in the limit $ka \to 0$. We should however keep in mind that spheres are special; for other shapes, (4.21) is not exact even in the quasi-static limit.

We emphasize that scattering is zero in the approximation (4.21). The first non-vanishing term in the expansion of Q_{sca} can be obtained at the next order of the approximation (that is, quadratic in κ); see Sec. 4.7 for second-order results and a more formal development of the Born series.

4.3 Scattering amplitude

We now use the first equation in (4.9) and the asymptotic form of (4.10b) to derive an expression for the scattered field far away from the target. Consider the geometry illustrated in Fig. 2. Here the origin of a reference frame, $O \in \mathbb{V}$, is located inside the target, and the region Ω is a sphere of radius R with the center at O. We can characterize the point of observation, $P \in \partial\Omega$, by the radius-vector $\mathbf{r}$ drawn from O to P. We can write $\mathbf{r} = R\mathbf{s}$, where $\mathbf{s}$ is a unit vector ($\mathbf{s} \cdot \mathbf{s} = 1$) specifying the scattering direction.

We further assume that R is large compared to the smallest radius of a sphere that contains $\mathbb{V}$ completely [5]. In other words, $R \gg |\mathbf{r}'|$ for any $\mathbf{r}' \in \mathbb{V}$. Under these assumptions, we can develop the far-field approximation for $\mathsf{G}_{\text{R}}(\mathbf{r}, \mathbf{r}'; \omega)$. The exact expression for $\mathsf{G}_{\text{R}}(\mathbf{r}, \mathbf{r}'; \omega)$ is given in (4.10b), and it contains the factor $|\mathbf{r} - \mathbf{r}'|$, which we will consider first. For $\mathbf{r}$ and $\mathbf{r}'$ restricted as shown in Fig. 2, we have

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[5] We will state this condition more precisely towards the end of this section.

$$|\mathbf{r} - \mathbf{r}'| = \sqrt{r^2 - 2\mathbf{r} \cdot \mathbf{r}' + (r')^2} = \sqrt{R^2 - 2R\mathbf{s} \cdot \mathbf{r}' + (r')^2} \approx R - \mathbf{s} \cdot \mathbf{r}' . \quad (4.22)$$

It is sufficient to use the first term in the above approximate expression in all algebraic functions in (4.10b), but the second term must be retained inside the exponent. This is so because the product kr' can be not small compared to unity even if $R \gg r'$. At the same time, the higher-order terms in the expansion (4.22) can always be neglected for sufficiently large R. We therefore proceed as follows. First, we keep only far-field contributions (those decaying as $1/|\mathbf{r} - \mathbf{r}'|$) in the expression (4.10b). Second, we replace all factors $\mathbf{r} - \mathbf{r}'$ by $\mathbf{r} = R\mathbf{s}$ in the algebraic functions, but use both terms of the approximation (4.22) inside the exponent. This results in the following asymptotic expression:

$$\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega) \sim k^2 \frac{e^{\mathrm{i}\,kR}}{R} \left(\mathsf{I}_3 - \mathbf{s} \otimes \mathbf{s} \right) e^{-\mathrm{i}\,k\mathbf{s}\cdot\mathbf{r}'} ; \quad \mathbf{r} = R\mathbf{s} \in \partial\Omega , \quad R \to \infty . \quad (4.23)$$

We next substitute this expression into (4.9), which results in

$$\boldsymbol{E}_{\mathrm{sca}}(\mathbf{s}R) \sim \boldsymbol{f}(\mathbf{s}) \frac{e^{\mathrm{i}\,kR}}{R} , \quad (4.24)$$

where

$$\boldsymbol{f}(\mathbf{s}) = k^2 \left(\mathsf{I}_3 - \mathbf{s} \otimes \mathbf{s} \right) \boldsymbol{P}(k\mathbf{s}) , \quad (4.25)$$

and we have introduced the spatial Fourier transform of the electric polarization field

$$\boldsymbol{P}(\mathbf{q}) = \int_{\mathbb{V}} \boldsymbol{P}(\mathbf{r}) e^{-\mathrm{i}\,\mathbf{q}\cdot\mathbf{r}} \mathrm{d}^3 r . \quad (4.26)$$

In the generic definition (4.26), $\mathbf{q}$ is arbitrary whereas in (4.25) it is restricted as $\mathbf{q} = k\mathbf{s}$ so that $|\mathbf{q}| = k$ (a constant for a fixed frequency). We say that the transform involved in (4.25) is evaluated on shell.

The vector function $\boldsymbol{f}(\mathbf{s})$ is known as the scattering amplitude. In the notations adopted above, the scattering amplitude depends on the direction of scattering, $\mathbf{s}$, which is included in the formal list of arguments. However, it also depends implicitly on the incident field via the factor $\boldsymbol{P}(k\mathbf{s})$, which, of course, also depends on the incident field. In Secs. 4.4.2 and 4.5, we will consider this dependence more explicitly.

To conclude this section, we note that, sufficiently far from the target, the scattered field is always a spherical wave of the form (4.24). This result may seem counter-intuitive. For example, a beam incident onto a small but macroscopic mirror is reflected according to the laws of geometrical optics, and the reflected field is not a spherical wave. The paradox can be resolved by noting that we have not quantified yet how far is "sufficiently far". Indeed, the radius R of the sphere $\partial\Omega$ should be large enough in order for the expansion (4.22) to yield an accurate approximation when used inside the exponent. To derive a quantitative condition for this to hold, we
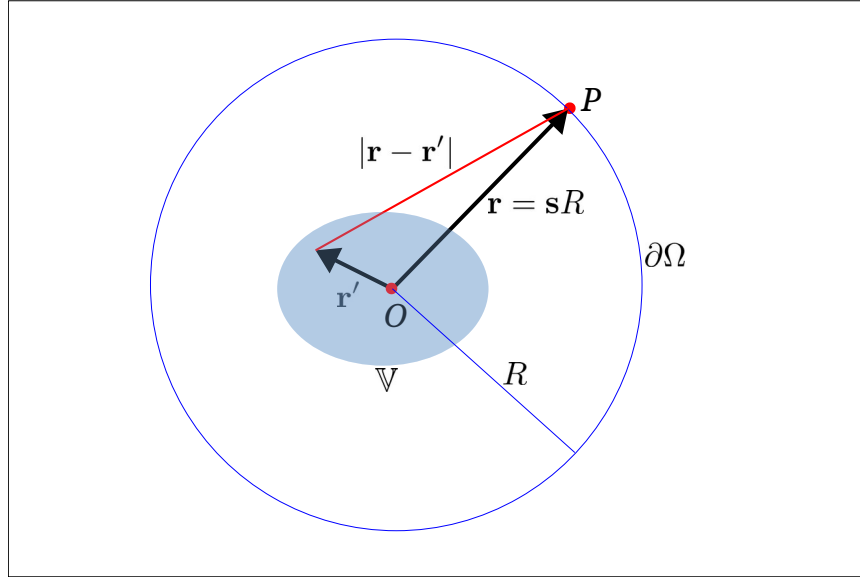
Fig. 2 Scattering geometry employed in the far-field approximation. Here $O \in \mathbb{V}$ is the origin of the reference frame and Ω is a spherical region of the radius R with the center at O.

note that the next term in the expansion (not shown in the formula) is of the order of $O((r')^2/R)$. Let the characteristic size of the target be D and therefore $(r')^2 \leq D^2$. We thus conclude that the next term in the expansion is of the order $O(D^2/R)$. Upon substitution into the exponent, this term is multiplied by k. Therefore, the expansion (4.22) truncated to the two first terms is an adequate approximation only if $kD^2/R \ll 1$. This means that the scattered field is a spherical wave only deep in Fraunhofer diffraction zone, that is, when $R \gg kD^2$. Consider the following example. For visible light of the wavelength $\lambda = 500$nm and a target with $D = 1$cm, the above condition is $R \gg 10^3$m. The inequality here must hold strongly. Such large distances are rarely considered in laboratory experiments. And at smaller distances, geometrical optics and then Fresnel diffraction theory are applicable. We thus see that the description in terms of the scattering amplitude and the associated theories of scattering are "natural" (in the sense that they agree with our intuitive expectations) only for targets that are not very large compared to the wavelength and, otherwise, only at very large distances. This observation is closely related to the so-called extinction paradox, which is considered in Sec. 6.4 below.

4.4 Optical theorem

Derivation of the optical theorem is often attributed to Van De Hulst (1949) [36] and Jones (1955) [37], although the basic ideas were fixed in the works of Sellmeier

and Rayleigh as early as in 1871, and then by Mie in 1908 for the special case of spherical particles. The relevant historical references can be found in [38]. It is not surprising that the optical theorem first came to the fore in Rayleigh's theory of the color of the sky since the phenomenon of extinction is closely related to wavelength-dependent attenuation of solar radiation in the atmosphere. The context in which the optical theorem was developed is quite complicated and involves phenomena that are outside of the scope of this chapter. It is worth mentioning however that one important question related to extinction, which was intensely debated in the beginning of 20-th century, was whether scattering from density fluctuations can account for the observed attenuation of light in the atmosphere [39]. From the contemporary point of view, the questions raised in this debate are addressed in the theories of electromagnetic homogenization and radiative transport in complex media.

The optical theorem has been originally derived for plane, monochromatic incident waves, and the problem was considered in this physical setting for a long time. For example, as late as in 1995, it came as a bit of a surprise that the optical theorem (in its conventional form) does not hold for sufficiently narrow Gaussian beams [40]. Various generalizations of the optical theorem to more complex incident fields were later developed [27, 41–44]. Still, the average textbook derivation of the theorem is quite convoluted; see for example [2], pp.71-73 where complicated three-dimensional integrals and a "considerable amount of algebraic manipulation" are involved. The complexity of these derivations grows from attempts to use the global (surface integral) definition of the extinguished power. Here we will show that the optical theorem is an elementary consequence of the local (volume integral) formulas (3.19) or (4.5c), which were derived above independently and without much effort.

4.4.1 Single plane wave

We start with the case of a single monochromatic, homogeneous plane wave. Assume that the incident field $\mathbf{E}_{\text{inc}}(\mathbf{r}, t)$ is of the form (4.2) with the complex amplitude

$$\boldsymbol{E}_{\text{inc}}(\mathbf{r}) = \boldsymbol{A} e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}, \quad \mathbf{k} = k\mathbf{u}, \quad k = \omega/c . \tag{4.27}$$

Here $\mathbf{u}$ is a real unit vector ($\mathbf{u} \cdot \mathbf{u} = 1$) specifying the direction of propagation and $\boldsymbol{A}$ is the wave amplitude. Some components of $\boldsymbol{A}$ can be complex depending on the state of polarization. Since $\mathbf{E}_{\text{inc}}(\mathbf{r}, t)$ satisfies the free-space Maxwell's equations (3.3), the wave (4.27) is transverse so that $\boldsymbol{A} \cdot \mathbf{u} = 0$. This relation holds even if $\boldsymbol{A}$ is complex. The time-averaged current of energy associated with the incident wave (4.27) is given by

$$\overline{\mathbf{S}}_{\text{inc}} = \frac{c}{8\pi}|A|^2 \, \mathbf{u} , \tag{4.28}$$

where $|A|^2 = A^* \cdot A$. The factor $c/8\pi$ (rather than $c/4\pi$, which is found in the expression for the Poynting vector) appears due to time averaging and is easy to trace. Note that, for a homogeneous [6] incident wave, $\overline{\mathbf{S}}_{\text{inc}}$ is independent of $\mathbf{r}$.

Now we substitute (4.27) into (4.5c), which results in

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} A^* \cdot P(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d^3r \ . \tag{4.29}$$

Recalling the definition of Fourier transform of the electric polarization field (4.26), we can re-write the above equation identically as

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \left[A^* \cdot P(\mathbf{k}) \right] \ . \tag{4.30}$$

Now consider the expression for the scattering amplitude evaluated *in the forward direction*. That is, let us replace the generic scattering direction $\mathbf{s}$ in (4.25) with $\mathbf{u}$:

$$f(\mathbf{u}) = k^2 \left(\mathbf{I}_3 - \mathbf{u} \otimes \mathbf{u} \right) P(\mathbf{k}) \ . \tag{4.31}$$

By dot-multiplying both sides of (4.31) by A^* and accounting for the condition $A^* \cdot \mathbf{u} = 0$ [7], we obtain the relation

$$A^* \cdot f(\mathbf{u}) = k^2 A^* \cdot P(\mathbf{k}) \ . \tag{4.32}$$

Comparing (4.32) to (4.30), we conclude that

$$Q_{\text{ext}} = \frac{\omega}{2k^2} \text{Im} \left[A^* \cdot f(\mathbf{u}) \right] \ . \tag{4.33}$$

To obtain a more conventional form of the optical theorem, we define the extinction cross section σ_{ext} as the ratio of the extinguished power and the incident energy flux. The latter is given by the scalar pre-factor in (4.28). Therefore,

$$\sigma_{\text{ext}} = \frac{4\pi}{k|A|^2} \text{Im} \left[A^* \cdot f(\mathbf{u}) \right] \ . \tag{4.34}$$

We thus have proved that the extinguished power is mathematically related to the scattering amplitude evaluated in just one – forward – direction. This result is quite remarkable. Note that it is specific to extinction. Neither the absorbed nor the scattered power can be similarly expressed in terms of $f(\mathbf{s})$ evaluated in a single direction $\mathbf{s} = \mathbf{u}$.

In the case of scattering, it is more convenient to use the global representations to express Q_{sca} in terms of $f(\mathbf{s})$. The global representation of Q_{sca} is given by (3.8) and the time-averaged energy current $\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r})$, which enters (3.8), is given by (4.3b). We thus need to know the amplitudes $E_{\text{sca}}(\mathbf{r})$ and $B_{\text{sca}}(\mathbf{r})$ for $\mathbf{r} \in \partial\Omega$. As above,

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[6] By this we mean that the wave is not evanescent or, equivalently, that $\mathbf{u}$ is a purely real vector.

[7] Since $\mathbf{u}$ is purely real, we have $(A \cdot \mathbf{u})^* = A^* \cdot \mathbf{u} = 0$.

we work in the asymptotic regime wherein $\partial\Omega$ is a sphere of a large radius R, as is illustrated in Fig. (2). In this case, $E_{\text{sca}}(\mathbf{r})$ at a point $\mathbf{r} = \mathbf{s}R \in \partial\Omega$ is given by the asymptotic formula (4.24). Since $\mathbf{s} \cdot \boldsymbol{f}(\mathbf{s}) = 0$, the scattered field is transverse, which means that $\mathbf{s} \cdot E_{\text{sca}}(\mathbf{s}R) = 0$. Same can be said about the magnetic field. For the latter, we do not need a detailed derivation. It follows from the symmetry of Maxwell's equations that $B_{\text{sca}}(\mathbf{s}R)$ is of the same magnitude as $E_{\text{sca}}(\mathbf{s}R)$ and is orthogonal to both $E_{\text{sca}}(\mathbf{s}R)$ and $\mathbf{s}$. Thus, $E_{\text{sca}}(\mathbf{s}R)$ and $B_{\text{sca}}(\mathbf{s}R)$ are of the same magnitude, mutually orthogonal, and lie in the tangential plane to $\partial\Omega$ touching the latter at the point $\mathbf{r} = \mathbf{s}R$. The vector product of $E_{\text{sca}}(\mathbf{s}R) \times B_{\text{sca}}(\mathbf{s}R)$ is therefore aligned with $\mathbf{s}$, which is also equal to the outward unit normal to $\partial\Omega$, $\mathbf{n}$. Putting everything together, we have

$$Q_{\text{sca}} = \frac{c}{8\pi} \int\limits_{4\pi} |\boldsymbol{f}(\mathbf{s})|^2 \mathrm{d}^2 s \, , \qquad (4.35)$$

where integration is over the entire solid angle; for example, in spherical coordinates (θ, ϕ), the element of a solid angle is $\mathrm{d}^2 s = \sin\theta \mathrm{d}\theta \mathrm{d}\phi$. Similarly to (4.34), we can normalize the scattered power to the incident flux to yield the scattering cross section, σ_{sca}, viz,

$$\sigma_{\text{sca}} = \frac{1}{|A|^2} \int\limits_{4\pi} |\boldsymbol{f}(\mathbf{s})|^2 \mathrm{d}^2 s \, . \qquad (4.36)$$

Comparing this result to (4.34), we see that the scattering amplitude for a plane wave illumination satisfies the inequality

$$\mathrm{Im}\,[A^* \cdot \boldsymbol{f}(\mathbf{u})] \geq \frac{k}{4\pi} \int\limits_{4\pi} |\boldsymbol{f}(\mathbf{s})|^2 \mathrm{d}^2 s \, . \qquad (4.37)$$

Equality in the above relation holds only for non-absorbing targets.

We do not adduce here any expressions for the absorbed power or cross section in terms of the scattering amplitude because these quantities can be expressed as differences between (4.33) and (4.35) or between (4.34) and (4.36), and no simpler expressions can be given without specifying the exact properties of the target.

4.4.2 Several plane waves

Now consider the case when the incident field consists of several homogeneous plane waves of arbitrary amplitudes and directions but oscillating at the same frequency ω. In this case, the complex amplitude of the incident electric field is given by

$$E_{\text{inc}}(\mathbf{r}) = \sum_n A_n e^{i\mathbf{k}_n \cdot \mathbf{r}} \, , \quad \mathbf{k}_n = k\mathbf{u}_n \, , \quad k = \omega/c \, . \qquad (4.38)$$

From the linearity of Maxwell's equations, we know that the electric polarization field excited by the incident wave (4.38) is a superposition of polarization fields that would have existed if only one of the plane waves was present. Mathematically, this is expressed by writing

$$P(\mathbf{r}) = \sum_n P_n(\mathbf{r}) \,, \tag{4.39}$$

where $P_n(\mathbf{r})$ is the polarization field that would have existed in the target if the incident field consisted only of the n-th plane wave. Using the volume-integral definition of the extinguished power (4.5c) and the Fourier transform definition (4.26), we find that

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \sum_{nm} \int_{\mathbb{V}} A_n^* \cdot P_m(\mathbf{r}) e^{-i \mathbf{k}_n \cdot \mathbf{r}} \, \mathrm{d}^3 r = \frac{\omega}{2} \text{Im} \sum_{nm} A_n^* \cdot P_m(\mathbf{k}_n) \,. \tag{4.40}$$

The form of this equation indicates how to proceed; however, we now need slightly more specialized notations. Recall that equation (4.25) defines the scattering amplitude for a generic incident field. Dependence on the latter is implicit in the function $P(k\mathbf{s})$ appearing in the right-hand side of (4.25) since $P(\mathbf{r})$ and its Fourier transform $P(\mathbf{q})$ depend on the incident field. To make this dependence explicit, we introduce a notation for the scattering amplitude for a single incident plane wave of the form $A_m e^{i \mathbf{k}_m \cdot \mathbf{r}}$, where $\mathbf{k}_m = \mathbf{u}_m k$, namely,

$$f(A_m, \mathbf{u}_m; \mathbf{s}) = k^2 (\mathbb{I}_3 - \mathbf{s} \otimes \mathbf{s}) \, P_m(k\mathbf{s}) \,. \tag{4.41}$$

Equation (4.41) is simply (4.25) re-written for a more specific illumination and using a more explicit notation. The interpretation of $f(A_m, \mathbf{u}_m; \mathbf{s})$ is the following: if we illuminate the target by the incident plane wave of the form $A_m e^{i k \mathbf{u}_m \cdot \mathbf{r}}$, the scattered field in the asymptotic region far away from the target would be given by

$$E_{\text{sca}}(R\mathbf{s}) = f(A_m, \mathbf{u}_m; \mathbf{s}) \frac{e^{i kR}}{R} \,. \tag{4.42}$$

Note that the scattering amplitude $f(A_m, \mathbf{u}_m; \mathbf{s})$ depends on both the polarization vector A_m and the propagation direction $\mathbf{u}_m$ of the incident plane wave.

   Now let us evaluate the scattering amplitude in the direction $\mathbf{s} = \mathbf{u}_n$ (the direction of one of the plane waves in the expansion (4.38), where n can coincide with m). Using the same arguments as above, we find that

$$A_n^* \cdot f(A_m, \mathbf{u}_m; \mathbf{u}_n) = k^2 A_n^* \cdot P_m(\mathbf{k}_n) \,. \tag{4.43}$$

But the expression in the right-hand side of (4.43) is exactly the term appearing in the decomposition of the extinguished power (4.40). We therefore conclude that

$$Q_{\text{ext}} = \frac{\omega}{2k^2} \text{Im} \sum_{nm} A_n^* \cdot f(A_m, \mathbf{u}_m; \mathbf{u}_n) \,. \tag{4.44}$$

This is the generalized optical theorem for several incident plane waves oscillating at the same frequency ω. Note that it involves the scattering amplitude with the incidence directions direction $\mathbf{u}_m$ and outgoing directions $\mathbf{u}_n$ for all combinations of the indexes n and m, which label the individual plane waves in the decomposition of the total incident field (4.38). Thus, n and m in (4.44) run over the same set of values.

The result (4.44) reinforces our previous observation that extinction is a complicated interference phenomenon. In particular, (4.44) suggests that the extinction is not additive in the sense that

$$Q_{\text{ext}} \neq \frac{\omega}{2k^2} \text{Im} \sum_n A_n^* \cdot f(A_n, \mathbf{u}_n; \mathbf{u}_n) \quad \text{(strictly monochromatic fields)}, \quad (4.45)$$

where the right hand side is a sum of extinguished powers that would have existed if only n-th incident plane wave was present (computed according to (4.33)). However, this conclusion holds only for strictly monochromatic fields, which, as we know, do not exist in nature. Then, what would happen if we account for a finite spectral width of the incident fields, however tiny? Will the off-diagonal terms survive? The answer to this question can be given by the theory of partial coherence, which is considered in Sec. 5 below. For now, we can say on physical grounds that the off-diagonal terms in (4.44) should indeed be neglected if the different plane waves in (4.38) are generated by physically-independent sources, i.e., by several different lasers. Then, no matter how monochromatic these lasers are, the off-diagonal terms will average to zero over sufficiently long periods of time. Correspondingly, the extinguished power is, in fact, additive in this case, and we can write

$$Q_{\text{ext}} = \frac{\omega}{2k^2} \text{Im} \sum_n A_n^* \cdot f(A_n, \mathbf{u}_n; \mathbf{u}_n) \quad \text{(independent sources)}. \quad (4.46)$$

If, on the other hand, the different incident waves in (4.38) are generated by one physical source with the use of beam splitters, mirrors, prisms and similar devices, then the off-diagonal terms in (4.44) should be retained. However, the assumption of source monochromaticity may in this case be insufficient and a more accurate theory will involve statistical properties of the incident field.

In the case of several incident plane waves, there is no well-defined forward direction and therefore extinction cross section does not have the same simple interpretation as in the case of a single plane wave. Still, we can formally divide (4.44) by the sum of the energy fluxes of all incident plane waves. We then obtain a quantity with a dimensionality of area,

$$\sigma_{\text{ext}} = \frac{4\pi}{k \sum_n |A_n|^2} \text{Im} \sum_{nm} A_n^* \cdot f(A_m, \mathbf{u}_m; \mathbf{u}_n). \quad (4.47)$$

This quantity is not a "true" cross section; it is not possible to identify a flat region of the area given by (4.47) such that the total incident energy flux through it is given by (4.44).

We can also adduce an expression for the scattered power although scattering experiments are rarely conducted using several incident plane waves. Nevertheless, the formal definitions of the scattered power remain applicable. Using the global definition (3.8) and proceeding similarly to the case of a single plane wave, we obtain

$$Q_{\text{sca}} = \frac{c}{8\pi} \sum_{nm} \int_{4\pi} \boldsymbol{f}^*(\boldsymbol{A}_m, \mathbf{u}_m; \mathbf{s}) \cdot \boldsymbol{f}(\boldsymbol{A}_n, \mathbf{u}_n; \mathbf{s}) \, \mathrm{d}^2 s \,, \qquad (4.48)$$

The off-diagonal terms in this summation account for the interference of the scattered fields produced by different incident plane waves. Just as in the case of extinction, the double summation is retained if the incident plane waves are produced by the same physical source. If the waves are physically independent, the off-diagonal terms in (4.48) can be suppressed as they do not yield physically-measurable contributions to the scattered power.

4.5 T-matrix and a symmetric form of the optical theorem

The mathematical form of the optical theorem derived by us so far is non-symmetric. This is especially obvious for the case of multiple plane waves. Indeed, expression (4.44) contains the terms $\boldsymbol{A}_n^* \cdot \boldsymbol{f}(\boldsymbol{A}_m, \mathbf{u}_m; \mathbf{u}_n)$ in the right-hand side. This is not very satisfying. One can ask: why does the amplitude $\boldsymbol{A}_n$ appear as a factor whereas $\boldsymbol{A}_m$ shows up in the list of arguments of $\boldsymbol{f}$? The answer is that we have not yet explored linearity of the underlying equations to the fullest extent and therefore formulated the optical theorem in terms of the scattering amplitude rather than in terms of the T-matrix. We will now fix this shortcoming.

We start by observing that, from linearity of (4.17), we can write the formal solution for the electric polarization field in the form

$$\boldsymbol{P}(\mathbf{r}) = \int_{\mathbb{V}} \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{E}_{\text{inc}}(\mathbf{r}') \mathrm{d}^3 r' \,. \qquad (4.49)$$

Here $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega)$ is the integral kernel of the frequency-domain T-matrix [8]. Importantly, the T-matrix is completely defined by the shape and properties of the target but independent of the incident field. This is a manifestation of the electromagnetic superposition principle. We state the equations from which the T-matrix can be computed below in (4.57) and (4.63).

Now let the incident field be a superposition of plane waves of the form (4.38) and $\boldsymbol{P}(\mathbf{r})$ be a superposition of corresponding solutions of the form (4.39). We have

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[8] The notation for the T-matrix (T) should not be confused with the notation for the interval over which time averaging is performed (T). The T-matrix is an operator acting in the Hilbert space of vector functions that are square-integrable in $\mathbb{V}$. It has a tensorial kernel. See Sec. 4.7 for a more formal development.

from (4.49):

$$P_m(\mathbf{r}) = \int_{\mathbb{V}} \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) A_m e^{i\,\mathbf{k}_m \cdot \mathbf{r}'} \mathrm{d}^3 r' \ . \qquad (4.50)$$

We wish to use this result to develop the expression (4.41) for $\boldsymbol{f}(\boldsymbol{A}_m, \mathbf{u}_m; \mathbf{s})$. To this end, we need to compute $\boldsymbol{P}_m(k\mathbf{s})$. Using the definition (4.26) and the formula (4.50), we find that

$$P_m(k\mathbf{s}) = \int_{\mathbb{V}} \mathrm{d}^3 r \ P_m(\mathbf{r}) e^{-i\,k\mathbf{s}\cdot\mathbf{r}} = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \ \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) A_m e^{i\,(\mathbf{k}_m\cdot\mathbf{r}' - k\mathbf{s}\cdot\mathbf{r})} \ .$$

$$(4.51)$$

It is convenient at this point to introduce the T-matrix in spatial Fourier representation, viz,

$$\mathsf{T}(\mathbf{q}, \mathbf{q}'; \omega) = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \ e^{-i\,\mathbf{q}\cdot\mathbf{r}} \ \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \ e^{i\,\mathbf{q}'\cdot\mathbf{r}'} \ . \qquad (4.52)$$

Here $\mathbf{q}$ and $\mathbf{q}'$ are generic Fourier variables. According to the general convention adopted in this chapter, we use the same letter for the T-matrix in the real-space and Fourier-space representations. The particular representation used will be indicated by the list of arguments. We can now re-write (4.51) as

$$P_m(k\mathbf{s}) = \mathsf{T}(k\mathbf{s}, \mathbf{k}_m; \omega) A_m \ . \qquad (4.53)$$

Substituting (4.53) into (4.41), we obtain

$$\boldsymbol{f}(\boldsymbol{A}_m, \mathbf{u}_m; \mathbf{s}) = k^2 \left(\mathsf{I}_3 - \mathbf{s} \otimes \mathbf{s} \right) \mathsf{T}(k\mathbf{s}, \mathbf{k}_m; \omega) \boldsymbol{A}_m \ . \qquad (4.54)$$

Finally, using this expression in (4.44) and accounting for $\boldsymbol{A}_n \cdot \mathbf{u}_n = A_n^* \cdot \mathbf{u}_n = 0$, we arrive at the symmetric form of the optical theorem for multiple incident plane waves:

$$Q_{\text{ext}} = \frac{\omega}{2} \operatorname{Im} \sum_{nm} \boldsymbol{A}_n^* \cdot \mathsf{T}(\mathbf{k}_n, \mathbf{k}_m; \omega) \boldsymbol{A}_m \ . \qquad (4.55)$$

We can divide this expression by the sum of all incident energy fluxes to obtain a quantity of the dimensionality of area:

$$\sigma_{\text{ext}} = \frac{4\pi k}{\sum_n |\boldsymbol{A}_n|^2} \operatorname{Im} \sum_{nm} \boldsymbol{A}_n^* \cdot \mathsf{T}(\mathbf{k}_n, \mathbf{k}_m; \omega) \boldsymbol{A}_m \ . \qquad (4.56)$$

It should be kept in mind though that σ_{ext} given by the above equation cannot be identified with any physical area. We also should keep in mind the conditions under which the off-diagonal terms in (4.56) yield physically measurable contributions. These conditions are discussed in Sec. 4.4.2 and then from a more fundamental

point of view in Sec. 5.5 below. If the conditions do not hold, we can drop the off-diagonal terms in (4.56) and the expression is simplified.

Thus, the extinguished power is a bilinear form of the plane wave amplitudes A_n with the coefficients given by a Fourier transform of the T-matrix. While $\mathsf{T}(\mathbf{q}, \mathbf{q}'; \omega)$ is defined for generic Fourier variables $\mathbf{q}$ and $\mathbf{q}$ according to (4.52), the optical theorem involves only a particular restriction of $\mathsf{T}(\mathbf{q}, \mathbf{q}'; \omega)$ with $\mathbf{q}, \mathbf{q}' \in \{\mathbf{k}_n : n = 1, 2, \ldots\}$. We say that the Fourier transform $\mathsf{T}(\mathbf{k}_n, \mathbf{k}_m; \omega)$ is on shell because the length of both arguments is fixed to the free-space wave number k.

It remains to state the equation from which $\mathsf{T}(\mathbf{k}_n, \mathbf{k}_m; \omega)$ can be computed. We first write the general integral equation, which defines the T-matrix in frequency domain. This equation can be obtained by substituting the ansatz (4.49) into (4.17), which results in

$$\mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) = \kappa(\mathbf{r}, \omega)\delta(\mathbf{r} - \mathbf{r}')\mathsf{I}_3 + \kappa(\mathbf{r}, \omega) \int_{\mathbb{V}} \mathrm{d}^3 r'' \; \mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}''; \omega)\mathsf{T}(\mathbf{r}'', \mathbf{r}'; \omega) \; .$$

$$(4.57)$$

It follows from (4.57) that the T-matrix possesses the following symmetry properties:

$$\mathsf{T}(\mathbf{r}, \mathbf{r}'; -\omega) = \mathsf{T}^*(\mathbf{r}, \mathbf{r}'; \omega) \; , \tag{4.58a}$$

$$\mathsf{T}(\mathbf{r}, \mathbf{r}'; \;\; \omega) = \mathsf{T}^{\mathrm{T}}(\mathbf{r}', \mathbf{r}; \omega) \; . \tag{4.58b}$$

The property (4.58a) is a reflection of the fact that $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega)$ is the temporal Fourier transform of the real-valued, time-domain T-matrix $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau)$, which is discussed in more detail in Sec. 5.3 below. The second property (4.58b) is the mathematical expression of the principle of electromagnetic reciprocity. Note that T is symmetric but not Hermitian.

Applying transformation (4.52) to (4.57), we obtain

$$\mathsf{T}(\mathbf{q}, \mathbf{q}'; \omega) = \kappa(\mathbf{q} - \mathbf{q}', \omega)\mathsf{I}_3$$
$$+ \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \int_{\mathbb{V}} \mathrm{d}^3 r'' \; \kappa(\mathbf{r}, \omega)\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}''; \omega)\mathsf{T}(\mathbf{r}'', \mathbf{r}'; \omega)e^{\mathrm{i}\,(\mathbf{q}' \cdot \mathbf{r}' - \mathbf{q} \cdot \mathbf{r})} \; , \quad (4.59)$$

where

$$\kappa(\mathbf{q}, \omega) = \int_{\mathbb{V}} \kappa(\mathbf{r}, \omega)e^{-\mathrm{i}\,\mathbf{q} \cdot \mathbf{r}} \, \mathrm{d}^3 r \; . \tag{4.60}$$

Expression (4.59) may seem to be difficult to develop further since plane waves do not generally form an orthogonal basis in $\mathbb{V}$. However, we can make progress by utilizing the translational invariance of $\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega)$. Since the latter depends only on $\mathbf{r} - \mathbf{r}'$ and not on $\mathbf{r}$ and $\mathbf{r}'$ separately, we can write

$$G_R(\mathbf{r}, \mathbf{r}'; \omega) = \int K_R(\mathbf{p}, \omega) e^{i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')} \frac{d^3 p}{(2\pi)^3} . \qquad (4.61)$$

Here integration is over the entire space. Expression for $K_R(\mathbf{p}, \omega)$ is rather simple [28]:

$$K_R(\mathbf{p}, \omega) = \frac{4\pi}{3} \frac{(2k^2 + p^2)I_3 - 3\mathbf{p} \otimes \mathbf{p}}{p^2 - k^2} . \qquad (4.62)$$

We now substitute the decomposition (4.61) into (4.59) and obtain

$$T(\mathbf{q}, \mathbf{q}'; \omega) = \kappa(\mathbf{q} - \mathbf{q}', \omega)I_3 + \int \kappa(\mathbf{q} - \mathbf{p}, \omega) K_R(\mathbf{p}, \omega) T(\mathbf{p}, \mathbf{q}'; \omega) \frac{d^3 p}{(2\pi)^3} . \quad (4.63)$$

Since $K_R(\mathbf{p}, \omega)$ is known analytically and $\kappa(\mathbf{q}, \omega)$ can be computed for any target according to (4.60), equation (4.63) defines $T(\mathbf{q}, \mathbf{q}'; \omega)$ completely. Unlike the volume integral equation (4.57), its Fourier-space counter-part (4.63) must be solved in the entire space (of $\mathbf{q}$). It helps however to know that $K_R(\mathbf{q}, \omega)$ is sharply peaked near $|\mathbf{q}| = k$. Therefore, numerically, the space of $\mathbf{q}$ can be truncated.

As before, we can iterate (4.63) and write the power-series expansion

$$T(\mathbf{q}, \mathbf{q}'; \omega) = \kappa(\mathbf{q} - \mathbf{q}', \omega)I_3 + \int \kappa(\mathbf{q} - \mathbf{p}, \omega) K_R(\mathbf{p}, \omega) \kappa(\mathbf{p} - \mathbf{q}', \omega) \frac{d^3 p}{(2\pi)^3} + \dots$$
$$(4.64)$$

First Born approximation is obtained by truncating the expansion after the first term. In this manner, we obtain a particularly simple form of the optical theorem:

$$Q_{ext} \approx \frac{\omega}{2} \text{Im} \sum_{nm} \kappa(\mathbf{k}_n - \mathbf{k}_m, \omega) \, \mathbf{A}_n^* \cdot \mathbf{A}_m . \qquad (4.65)$$

It can be seen that the extinguished power in the first Born approximation is sensitive to various spatial Fourier components of $\kappa(\mathbf{r}, \omega)$. The Fourier variable $\mathbf{k}_n - \mathbf{k}_m = k(\mathbf{u}_m - \mathbf{u}_n)$ is restricted to the ball of radius $2k$ centered at the origin and known as the Ewald sphere . The finite radius of Ewald sphere sets the resolution limit of the linearized inverse scattering problem (that is, determining properties of the target from measurements of the scattering amplitude).

4.6 Single plane wave

The case of a single incident plane wave deserves a separate consideration. We have already discussed this case in Sec. 4.4.1 above. However, the mathematical formalism of T-matrix offers us convenient tools to make further progress. We assume illumination of the form (4.27) and will work with the extinguished power Q_{ext}; the corresponding cross section σ_{ext} can be obtained as $\sigma_{ext} = 8\pi Q_{ext}/c|A|^2$. Specializing the general result (4.55) to a single plane wave, we obtain

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \left[\boldsymbol{A}^* \cdot \mathsf{T}(\mathbf{k}, \mathbf{k}; \omega) \boldsymbol{A} \right] . \tag{4.66}$$

While this formula is useful in its own right, we will write out the spatial Fourier transform of the T-matrix (according to (4.52)) and the algebraic operations in (4.66) as

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \sum_{ij} A_i^* A_j e^{\mathrm{i} \mathbf{k} \cdot (\mathbf{r}' - \mathbf{r})} \, T_{ij}(\mathbf{r}, \mathbf{r}'; \omega) . \tag{4.67}$$

Let

$$w_{ij}(\mathbf{r}, \mathbf{r}') = w_{ij}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}') + \mathrm{i} \; w_{ij}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}') = A_i^* A_j e^{\mathrm{i} \mathbf{k} \cdot (\mathbf{r}' - \mathbf{r})} . \tag{4.68}$$

Here the superscripts (r) and (i) indicate the real and imaginary parts. It can be seen that $w_{ij}(\mathbf{r}, \mathbf{r}')$ is a Hermitian kernel, that is, $w_{ij}(\mathbf{r}, \mathbf{r}') = w_{ij}^*(\mathbf{r}', \mathbf{r})$. In fact, $w_{ij}(\mathbf{r}, \mathbf{r}')$ is closely related to the cross spectral density of the incident field, which has the same property (see Eqs. (7.28) through (7.30) in Sec. 7.2.2 below). Using the notation introduced in (4.68), we can re-write (4.67) as

$$Q_{\text{ext}} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \sum_{ij} w_{ij}(\mathbf{r}, \mathbf{r}') \, T_{ij}(\mathbf{r}, \mathbf{r}'; \omega) . \tag{4.69}$$

Now, an important observation is that, from the Hermiticity of $w_{ij}(\mathbf{r}, \mathbf{r}')$, it follows that $w_{ij}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}')$ is symmetric and $w_{ij}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}')$ is anti-symmetric, that is,

$$w_{ji}^{(\mathrm{i})}(\mathbf{r}', \mathbf{r}) = -w_{ij}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}') . \tag{4.70}$$

On the other hand, according to (4.58b), the T-matrix is symmetric. Consequently, the term with $w_{ij}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}')$ in (4.69) is zero. We therefore obtain

$$Q_{\text{ext}} = \frac{\omega}{2} \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \sum_{ij} w_{ij}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}') \, T_{ij}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega) . \tag{4.71}$$

Restoring original notations, we obtain

$$Q_{\text{ext}} = \frac{\omega}{2} \, \boldsymbol{A}^* \cdot \mathsf{T}_{\text{ext}}(\mathbf{k}, \mathbf{k}; \omega) \boldsymbol{A} , \tag{4.72}$$

where

$$\mathsf{T}_{\text{ext}}(\mathbf{k}, \mathbf{k}; \omega) = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \, e^{-\mathrm{i} \mathbf{k} \cdot \mathbf{r}} \, \mathsf{T}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega) \, e^{\mathrm{i} \mathbf{k} \cdot \mathbf{r}'} . \tag{4.73}$$

Note that, although $\mathsf{T}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega)$ is real by definition, $\mathsf{T}_{\text{ext}}(\mathbf{k}, \mathbf{k}; \omega)$ is complex Hermitian. For this reason, expression (4.72) defines a real-valued quantity and no additional operation of taking the real part is necessary.

We thus have shown that it is the imaginary part of the real-space T-matrix that gives rise to extinction. The real part is unimportant. The notation introduced in (4.73) is a reflection of this observation: it indicates that $T_{ext}(\mathbf{k}, \mathbf{k}; \omega)$ is the spatial Fourier transform of the imaginary part of the T-matrix, which is responsible for extinction. Note however that both real and imaginary parts of $T_{ext}(\mathbf{k}, \mathbf{k}; \omega)$ are important because the amplitude $\mathbf{A}$ appearing in the expression (4.72) can be complex, i.e., for a circularly-polarized wave. A generalization of the result (4.72) is given in Sec. 4.7 below (Eq. (4.87)).

4.7 General results and formal relations

We have shown that the T-matrix appears naturally in the formulation of the optical theorem. However, the latter requires that there exists some "forward direction" for either one or several plane waves. We will now show that the T-matrix is also useful for general incident fields even when the forward direction cannot be defined.

We start with the local representation (4.5) in which the polarization field $\mathbf{P}(\mathbf{r})$ appears explicitly and express $\mathbf{P}(\mathbf{r})$ in terms of $\mathbf{E}_{inc}(\mathbf{r})$ using (4.49). This results in the expressions

$$Q_{abs} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} d^3r \int_{\mathbb{V}} d^3r' \ \mathbf{E}^*(\mathbf{r}) \cdot \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{E}_{inc}(\mathbf{r}') \ , \qquad (4.74a)$$

$$Q_{sca} = -\frac{\omega}{2} \text{Im} \int_{\mathbb{V}} d^3r \int_{\mathbb{V}} d^3r' \ \mathbf{E}^*_{sca}(\mathbf{r}) \cdot \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{E}_{inc}(\mathbf{r}') \ , \qquad (4.74b)$$

$$Q_{ext} = \frac{\omega}{2} \text{Im} \int_{\mathbb{V}} d^3r \int_{\mathbb{V}} d^3r' \ \mathbf{E}^*_{inc}(\mathbf{r}) \cdot \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \mathbf{E}_{inc}(\mathbf{r}') \ . \qquad (4.74c)$$

Thus, the absorption, scattering and extinction powers are matrix elements of the T-matrix, viewed as an integral operator. This can be formalized by writing

$$Q_{abs} = \frac{\omega}{2} \text{Im} \langle E|T(\omega)|E_{inc}\rangle \ , \qquad (4.75a)$$

$$Q_{sca} = -\frac{\omega}{2} \text{Im} \langle E_{sca}|T(\omega)|E_{inc}\rangle \ , \qquad (4.75b)$$

$$Q_{ext} = \frac{\omega}{2} \text{Im} \langle E_{inc}|T(\omega)|E_{inc}\rangle \ . \qquad (4.75c)$$

Here $T(\omega)$ is a linear operator acting on the Hilbert space $\mathcal{H}(\mathbb{V})$ of square-integrable vector function supported in $\mathbb{V}$, and $|E_{inc}\rangle$, $|E_{sca}\rangle$ and $|E\rangle = |E_{inc}\rangle + |E_{sca}\rangle$ are elements of $\mathcal{H}(\mathbb{V})$.

In (4.75), only Q_{ext} is given by a diagonal matrix element. Also, Q_{ext} seems to be the quantity most amenable to computations. Indeed, (4.75c) involves only $|E_{inc}\rangle$, which we always assume to be known, and $T(\omega)$, which characterizes the target at the frequency ω but is independent of $|E_{inc}\rangle$. Expressions (4.75a) and (4.75b) for

Q_{abs} and Q_{sca} appear to be more complicated as they contain, in addition, the total and the scattered fields. This complication is however not conceptual as we will demonstrate next.

We start with the scattered power. Recall that the T-matrix gives the polarization field $\boldsymbol{P}(\mathbf{r})$ in terms of the incident field $\boldsymbol{E}_{\mathrm{inc}}(\mathbf{r})$ according to (4.49). This can be written in the operator notations as $|\mathrm{P}\rangle = \mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\rangle$. Additionally, the Green's tensor gives the scattered field in terms of the polarization according to (4.9). Restricting the point $\mathbf{r}$ to $\mathbb{V}$ (equation (4.9) is valid in the larger region $\mathbb{D}$), we can write this relation as $|\mathrm{E}_{\mathrm{sca}}\rangle = \mathrm{G}(\omega)|\mathrm{P}\rangle$ [9]. Combining the two equations, we obtain the linear relation between the incident and scattered fields: $|\mathrm{E}_{\mathrm{sca}}\rangle = \mathrm{G}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\rangle$. Now, substituting this into (4.75b), we obtain the following expression for Q_{sca}:

$$
\begin{aligned}
Q_{\mathrm{sca}} &= -\frac{\omega}{2}\mathrm{Im}\left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{G}^{\dagger}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle \\
&= \frac{\omega}{2}\mathrm{Im}\left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{G}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle \\
&= \frac{\omega}{2}\mathrm{Im}\left[-\frac{4\pi}{3}\left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle + \left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{G}_{\mathrm{R}}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle\right] \\
&= \frac{\omega}{2}\mathrm{Im}\left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{G}_{\mathrm{R}}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle
\end{aligned}
\tag{4.76}
$$

where $\dagger$ denotes Hermitian conjugation of an operator. In the last two lines of this chain of equalities, we decomposed $\mathrm{G}(\omega)$ into singular and regular parts according to (4.10a) and then noted that $\mathrm{Im}\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}^{\dagger}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\rangle = 0$. It can be seen that Q_{sca} is also a diagonal element, but of a different operator, $\Pi_{\mathrm{sca}}(\omega) = \mathrm{T}^{\dagger}(\omega)\mathrm{G}_{\mathrm{R}}(\omega)\mathrm{T}(\omega)$. Thus, the scattered power can also be expressed in terms of the incident field and the T-matrix. It is true that the definition of $\Pi_{\mathrm{sca}}(\omega)$ involves, in addition, $\mathrm{G}_{\mathrm{R}}(\omega)$, but this operator is known analytically and has a simple form.

The absorbed power can be obtained by subtracting Q_{sca} from Q_{ext}, which results in

$$
Q_{\mathrm{abs}} = \frac{\omega}{2}\mathrm{Im}\left\langle \mathrm{E}_{\mathrm{inc}}|\mathrm{T}(\omega) - \mathrm{T}^{\dagger}(\omega)\mathrm{G}_{\mathrm{R}}(\omega)\mathrm{T}(\omega)|\mathrm{E}_{\mathrm{inc}}\right\rangle .
\tag{4.77}
$$

To obtain a more symmetric expression, we will make use of the explicit form of the T-matrix. It follows from (4.57) that

$$
\mathrm{T}(\omega) = \left[\mathrm{I}_{\mathbb{V}} - \mathrm{V}(\omega)\mathrm{G}_{\mathrm{R}}(\omega)\right]^{-1}\mathrm{V}(\omega) = \mathrm{V}(\omega)\left[\mathrm{I}_{\mathbb{V}} - \mathrm{G}_{\mathrm{R}}(\omega)\mathrm{V}(\omega)\right]^{-1} ,
\tag{4.78}
$$

where $\mathrm{I}_{\mathbb{V}}$ is the unit operator in $\mathcal{H}[\mathbb{V}]$, $\mathrm{V}(\omega)$ is defined by

$$
\mathrm{V}(\mathbf{r}, \mathbf{r}'; \omega) = \kappa(\mathbf{r}, \omega)\mathrm{I}_3\delta(\mathbf{r} - \mathbf{r}')
\tag{4.79}
$$

and $\kappa(\mathbf{r}, \omega)$ is the coupling function introduced in (4.18). The two expressions in (4.78) are equivalent. However, it might be easier to see the transitions when using one of forms of $\mathrm{T}(\omega)$. In particular, it follows immediately from the second expression

---

[9] Here $\mathrm{G}(\omega)$ includes both the singular and the regular parts, $\mathrm{G}(\omega) = -\frac{4\pi}{3}\mathrm{I}_{\mathbb{V}} + \mathrm{G}_{\mathrm{R}}(\omega)$.

in (4.78) that

$$G_R(\omega)T(\omega) = [I_V - G_R(\omega)V(\omega)]^{-1} - I_V \ . \tag{4.80}$$

This can be verified by replacing $G_R(\omega)$ with $G_R(\omega) - I_V + I_V$ in the left-hand side of (4.80) and then using the second form in (4.78) for $T(\omega)$. We therefore have

$$Q_{abs} = \frac{\omega}{2}\mathrm{Im}\left\langle E_{inc}|T(\omega) + T^\dagger(\omega) - T^\dagger(\omega)[I_V - G_R(\omega)V(\omega)]^{-1}|E_{inc}\right\rangle \ . \tag{4.81}$$

The term $\mathrm{Im}\langle E_{inc}|T(\omega) + T^\dagger(\omega)|E_{inc}\rangle$ obviously vanishes, and we have

$$Q_{abs} = -\frac{\omega}{2}\mathrm{Im}\left\langle E_{inc}|T^\dagger(\omega)[I_V - G_R(\omega)V(\omega)]^{-1}|E_{inc}\right\rangle \ . \tag{4.82}$$

Let us define the scattering operator

$$\Sigma(\omega) = [I_V - G_R(\omega)V(\omega)]^{-1} \ . \tag{4.83}$$

By comparing to (4.78), we see that $T(\omega) = V(\omega)\Sigma(\omega)$. Using this definition in (4.82), we find that

$$Q_{abs} = \frac{\omega}{2}\mathrm{Im}\left\langle E_{inc}|\Sigma^\dagger(\omega)V(\omega)\Sigma(\omega)|E_{inc}\right\rangle \ . \tag{4.84}$$

This is the symmetric form we were looking for. It can be easily seen that, if $V^\dagger(\omega) = V(\omega)$, then $Q_{abs} = 0$.

We thus have the following general relations:

$$Q_* = \frac{\omega}{2}\mathrm{Im}\left\langle E_{inc}|\Pi_*(\omega)|E_{inc}\right\rangle \ , \tag{4.85}$$

where $*$ stands for either abs, sca or ext, and

$$\Pi_{abs}(\omega) = \Sigma^\dagger(\omega)V(\omega)\Sigma(\omega) \ , \tag{4.86a}$$

$$\Pi_{sca}(\omega) = \Sigma^\dagger(\omega)V^\dagger(\omega)G_R(\omega)V(\omega)\Sigma(\omega) = T^\dagger(\omega)G_R(\omega)T(\omega) \ , \tag{4.86b}$$

$$\Pi_{ext}(\omega) = V(\omega)\Sigma(\omega) = T(\omega) \ . \tag{4.86c}$$

Note that only $\Pi_{ext}(\omega)$ is a symmetric operator whereas $\Pi_{abs}(\omega)$ and $\Pi_{sca}(\omega)$ are not. We can, however, re-write the above expressions identically by moving the operation of imaginary part inside the bracket as

$$Q_* = \frac{\omega}{2}\left\langle E_{inc}|H_*(\omega)|E_{inc}\right\rangle \ , \tag{4.87}$$

where

$$H_{abs}(\omega) = \Sigma^\dagger(\omega)V^{(i)}(\omega)\Sigma(\omega) \ , \tag{4.88a}$$

$$H_{sca}(\omega) = T^\dagger(\omega)G_R^{(i)}(\omega)T(\omega) \ , \tag{4.88b}$$

$$\mathsf{H}_{\text{ext}}(\omega) = \mathsf{T}^{(\text{i})}(\omega) \, , \tag{4.88c}$$

where the superscript (i) indicates imaginary part. Here all operators $\mathsf{H}_*(\omega)$ are Hermitian and $\mathsf{H}_{\text{ext}}(\omega)$ is real symmetric. For this reason (4.87) defines a real-valued quantity and can be viewed as a generalization of the result (4.72) derived previously for an incident plane wave.

In the case of a non-absorbing target, we have $\mathsf{V}^{(\text{i})}(\omega) = 0$ and, consequently, $Q_{\text{abs}} = 0$. The extinguished and scattered powers in this case are the same. This indeed follows from (4.88b) and (4.88c) as we have

$$\mathsf{T}^{(\text{i})}(\omega) = \mathsf{T}^{\dagger}(\omega) \, \mathsf{G}_{\mathsf{R}}^{(\text{i})}(\omega) \, \mathsf{T}(\omega) \quad \text{if } \mathsf{V}^{(\text{i})}(\omega) = 0 \, . \tag{4.89}$$

The easiest way to prove this identity is to compare the formal power expansions of each operator term-by-term keeping in mind that, in the considered case, V is real symmetric.

We have therefore expressed all powers of interest in terms of the incident field and the scattering operator, which is closely related to the T-matrix. It should be noted that computing the scattering operator is the most demanding part of solving the scattering problem. However, this step is independent of the form of the incident field. Therefore, once $\Sigma(\omega)$ is known, the solution for any incident field can be found quickly, with the most complex operation being a matrix-vector multiplication. In a practical implementation, a suitably truncated basis of functions in $\mathbb{V}$ is used to represent $|\mathsf{E}_{\text{inc}}\rangle$ as a vector of length N and $\Sigma(\omega)$ as an $N \times N$ matrix. If $\Sigma(\omega)$, the complexity of finding Q_* scales as $O(N^2)$ for any incident field. However, the complexity of computing $\Sigma(\omega)$ scales as $O(N^3)$. If N is too large, the complexity of computing $\Sigma(\omega)$ may become prohibitive. In such cases, methods are used that compute the field or polarization inside the target iteratively. The downside of such methods is that the computationally-intensive part must be repeated for any new incident field.

We finally adduce expressions for the Π-operators to second order in $\mathsf{V}(\omega)$:

$$\Pi_{\text{abs}}(\omega) \approx \mathsf{V}(\omega) + \mathsf{V}(\omega)\mathsf{G}_{\mathsf{R}}(\omega)\mathsf{V}(\omega) + \mathsf{V}^{\dagger}(\omega)\mathsf{G}_{\mathsf{R}}^{\dagger}(\omega)\mathsf{V}(\omega) \, , \tag{4.90a}$$

$$\Pi_{\text{sca}}(\omega) \approx \mathsf{V}^{\dagger}(\omega)\mathsf{G}_{\mathsf{R}}(\omega)\mathsf{V}(\omega) \, , \tag{4.90b}$$

$$\Pi_{\text{ext}}(\omega) \approx \mathsf{V}(\omega) + \mathsf{V}(\omega)\mathsf{G}_{\mathsf{R}}(\omega)\mathsf{V}(\omega) \, . \tag{4.90c}$$

It can be seen that, to first order in V, scattering is zero and extinction is equal to absorption. The approximation in which only linear terms in $\mathsf{V}(\omega)$ are kept is known as the first Born approximation and it was already considered by us in Sec. 4.2. Compared to the formulation based on the integral equations, the form (4.85) makes it much easier to obtain higher-order approximations.

4.8 Discrete spectrum

A straightforward generalization to the strictly monochromatic case is the discrete spectrum. We can introduce an incident field containing several strictly monochromatic components in a manner similar to (4.2), viz,

$$\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \text{Re} \sum_{\mu=1}^{N} \boldsymbol{E}_\mu(\mathbf{r}) e^{-\mathrm{i}\,\omega_\mu t} \,. \tag{4.91}$$

Here $\{\omega_\mu\}$ is an ordered discrete set of N positive frequencies, $0 < \omega_1 < \omega_2 < \ldots < \omega_N$, and we require that

$$\min_\mu (\omega_{\mu+1} - \omega_\mu) \geq \delta > 0 \,. \tag{4.92}$$

The spatial and temporal properties of the field (4.91) can be quite complicated. In particular, it does not have a single position-independent spectrum. We also cannot assign to this field a state of polarization. Indeed, the electric field vector evaluated at a given point $\mathbf{r}$ according to (4.91) is not generally confined to a plane [10] but rather moves quasi-chaotically in three dimensions.

Under some additional conditions [11], the field (4.91) is stationary in the wide sense [45, Sec.2.2.3]. In what follows, we will show that (4.91) satisfies the stationarity condition as it was formulated in Sec. 3.1 above, namely, that the energy-related quantities of interest, when averaged over the time window $[t - T/2, t + T/2]$ with $T \gg 1/\delta$, are independent of t. We note that such averaging is possible only if (4.92) holds and, therefore, this condition is important for stationarity.

It follows from the linearity of Maxwell's equations that each spectral component in (4.91) will excite in the target an electric polarization field oscillating at the same frequency and, therefore, of the form $\text{Re}[\boldsymbol{P}_\mu(\mathbf{r}) e^{-\mathrm{i}\,\omega_\mu t}]$. According to (3.5), the corresponding component of the induced current is $\text{Re}[\boldsymbol{J}_\mu(\mathbf{r}) e^{-\mathrm{i}\,\omega_\mu t}]$ with $\boldsymbol{J}_\mu = -\mathrm{i}\,\omega_\mu \boldsymbol{P}_\mu(\mathbf{r})$. The total polarization field and induced current are given by the superpositions of such terms,

$$\mathbf{P}(\mathbf{r}, t) = \text{Re} \sum_\mu \boldsymbol{P}_\mu(\mathbf{r}) e^{-\mathrm{i}\,\omega_\mu t} \,, \quad \mathbf{J}_{\text{ind}}(\mathbf{r}, t) = \text{Im} \sum_\mu \omega_\mu \boldsymbol{P}_\mu(\mathbf{r}) e^{-\mathrm{i}\,\omega_\mu t} \,. \tag{4.93}$$

Each spectral component in this decomposition satisfies its own integral equation of the form (4.17) but with all frequency-dependent quantities evaluated at $\omega = \omega_\mu$, viz,

---

[10] In the monochromatic case, the electric field vector evaluated at any given point in space as a function of time is always confined to a plane and moves along an ellipse (a straight line segment or a circle in the extreme cases of linear and circular polarizations).

[11] These conditions mainly concern avoiding or pushing to infinity the special space-time points. For example, if there is a point in space $\mathbf{r}_0$ such that $\boldsymbol{E}_\mu(\mathbf{r}_0) > 0$ for all μ, then $(\mathbf{r} = \mathbf{r}_0, t_0 = 0)$ is a special space-time point where the field has a sharp maximum. However, special points do not affect time-averaged power measurements, at least, not in the limit $T \to \infty$.

$$P_\mu(\mathbf{r}) = \kappa(\mathbf{r}, \omega_\mu) \left[E_\mu(\mathbf{r}) + \int_{\mathbb{V}} \mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \omega_\mu) P_\mu(\mathbf{r}') \mathrm{d}^3 r' \right] ; \quad \mathbf{r} \in \mathbb{V} . \quad (4.94)$$

We can now compute the extinguished power Q_{ext} by time averaging according to the volume-integral definitions (3.19). Substituting the spectral decompositions (4.93) for $\mathbf{J}_{\mathrm{ind}}(\mathbf{r}, t)$ and (4.91) for $\mathbf{E}_{\mathrm{inc}}(\mathbf{r}, t)$ into (3.19), we obtain

$$Q_{\mathrm{ext}} = \mathrm{Im} \sum_{\mu,\nu} \frac{\omega_\mu}{2} \int_{\mathbb{V}} \left[P_\mu(\mathbf{r}) \cdot E_\nu(\mathbf{r}) \langle e^{-\mathrm{i}(\omega_\mu + \omega_\nu)t} \rangle \right.$$
$$\left. + P_\mu(\mathbf{r}) \cdot E_\nu^*(\mathbf{r}) \langle e^{-\mathrm{i}(\omega_\mu - \omega_\nu)t} \rangle \right] \mathrm{d}^3 r . \quad (4.95)$$

The time averages in the above expression can be easily evaluated :

$$\langle e^{-\mathrm{i}(\omega_\mu \pm \omega_\nu)t} \rangle = \frac{1}{T} \int_{-T/2}^{T/2} e^{-\mathrm{i}(\omega_\mu \pm \omega_\nu)(t+\tau)} \mathrm{d}\tau$$
$$= \frac{2}{T} \frac{\sin[(\omega_\mu \pm \omega_\nu)T/2]}{\omega_\mu \pm \omega_\nu} e^{-\mathrm{i}(\omega_\mu \pm \omega_\nu)t} . \quad (4.96)$$

All averages of the form (4.96) with the minus sign and $\mu = \nu$ are equal to 1. The rest are of the order of $O((\delta T)^{-1})$ or smaller. Since we assume that $T \gg 1/\delta$, these terms are small compared to unity. It is true that there are many such small terms in the double summation (4.95), but these terms have random signs and are expected to average to zero. In any event, these "off-diagonal" terms uniformly go to zero when $T \to \infty$, and their sum can be made arbitrarily small. We therefore conclude that, with sufficient averaging, only the "diagonal" terms contribute significantly to (4.95). The remainder of the sum is not strictly zero but constitutes an unavoidable noise, which, in a practical measurement, can be suppressed by increasing the measurement time.

Keeping only the dominating terms in (4.95), we obtain

$$Q_{\mathrm{ext}} = \sum_\mu \frac{\omega_\mu}{2} \mathrm{Im} \int_{\mathbb{V}} E_\mu^*(\mathbf{r}) \cdot P_\mu(\mathbf{r}) \, \mathrm{d}^3 r = \sum_\mu Q_{\mathrm{ext}}^{(\mu)} , \quad (4.97\mathrm{a})$$

where, in the second equality, we have used equation (4.5c). According to this equation, $Q_{\mathrm{ext}}^{(\mu)}$ (one term in the summation in (4.97a)) is the extinguished power that would have existed if the target was illuminated by the strictly monochromatic incident field $\mathrm{Re}[E_\mu e^{-\mathrm{i}\omega_\mu t}]$. Formulas for the latter quantity (in both global and local representation) have been derived in Sec. 4 above. In particular, (4.5c) is the local representation for $Q_{\mathrm{ext}}^{(\mu)}$. Expressions for absorption and scattering can be obtained in exactly the same manner:

$$Q_{\text{abs}} = \sum_{\mu} Q_{\text{abs}}^{(\mu)} , \quad Q_{\text{sca}} = \sum_{\mu} Q_{\text{sca}}^{(\mu)} . \qquad (4.97b)$$

Thus, every isolated monochromatic spectral component of the incident field is absorbed, scattered and extinguished independently. If the spectrum is discrete, we can solve the electromagnetic problem at any isolated frequency assuming that other frequencies do not exist, compute Q_{ext}, Q_{abs} and Q_{sca}, and then just add the results computed at each frequency together according to (4.97).

There are some implications of (4.97) that we should consider with caution. As was mentioned in Sec. 4.4.2, realistic incident fields are not strictly monochromatic but partially coherent with possibly narrow but continuous spectra. The result that different spectral components are extinguished independently remains valid if the spectra of these components do not overlap significantly. However, if such overlap exists, the relevant consideration is whether the different components are generated by physically-independent sources or, perhaps, by the same source whose output is then split using beam splitters, mirrors and similar devices. In the former case, the conclusion that extinction is additive remains valid. In the latter case, we must consider explicitly the statistical properties of the incident field, and the conclusion of spectral additivity (at least, in its naive form) may no longer hold. While in many cases the intuitive idea of spectral additivity is correct (this includes, notably, the case of an incident plane wave with many spectral components), the general formula applicable to a partially-coherent incident field is more complicated. Even in the simpler cases, when the additivity apparently works, the spectrum of the incident field must be defined in a statistical sense and not as a Fourier transform. The relevant theory is discussed in more detail below.

5 Partially-coherent fields

5.1 Motivation and review

Effects of partial coherence of the incident field on extinction, absorption and scattering have been rarely considered in the literature until 1990-ies. There are many reasons for this. Perhaps, the most important reason is that detailed consideration of coherence is unnecessary in a typical application, which involves incident plane waves and relatively small targets. Under these conditions, one can safely rely on the intuitive idea of spectral additivity. To cite one of the early precursors of the theory discussed below, "We consider here one frequency component in the partially coherent field. This may be taken as a quasi-monochromatic field or used to produce a full polychromatic description in the usual way" (Carpenter and Pask [46]). By "usual way" the authors of [46] mean using the quantities computed for one frequency in a spectral integral. For example, we can compute the extinction cross section of a sphere $\sigma_{\text{ext}}(\omega)$ using the Mie theory for monochromatic fields, multiply it by the spectral density of the incident wave $S_{\text{inc}}(\omega)$ (defined in Sec. 5.2 below),

and integrate the product over frequencies. For non-spherical targets, one may need to keep track of polarization, but this is accomplished with simple algebra.

In many practically-important problems, there is nothing wrong with this approach. In some other cases, one needs to develop a more nuanced theory. As we will show in Sec. 5.4, the extinguished, absorbed and scattered powers can be expressed as spectral integrals under most general conditions. However, these integrals contain the frequency-domain T-matrix or the relevant Π-operators of the target (defined in (4.86) above). These mathematical objects are significantly more complicated than optical cross sections or polarizabilities of small particles. Only under special conditions the spectral integrals can be reduced to a form that contains the latter quantities. If by the "usual way" we understand integrating the product $\sigma_{\texttt{ext}}(\omega)S_{\texttt{inc}}(\omega)$ over frequencies, then this approach is not *always* correct.

The above difficulty, either directly or indirectly, has motivated a number of investigations in which statistical properties of the incident field were accounted for explicitly [47–54]. Some of these papers derive general results such as the optical theorem for partially-coherent fields [47, 48, 50] and others perform computations for spherical targets [49, 51, 53, 54] or for more complicated shapes [52]. A common feature of these works is that they consider physical situations in which the quantity of interest is not reducible to a simple spectral integral of $\sigma_{\texttt{ext}}(\omega)S_{\texttt{inc}}(\omega)$ or of a similar combination. More specifically, the mentioned references investigate illumination by quasi-monochromatic Gaussian Schell-model beams [55, 56]. These fields are different from plane waves and, unlike the latter, are characterized by a finite transverse coherence length. Spectral width is not introduced explicitly and statistical properties of the incident field are characterized in space-frequency domain using the mathematical technique developed by Wolf in 1980-ies [10–12]. It is however understood that, in the case of polychromatic fields, power-related quantities can be obtained by spectral integration.

The goal of this section is not to reproduce the results of the above references but to describe a general theoretical framework in which all such results can be obtained as special cases. The T-matrix of the target and, more generally, the Π-operators introduced in Sec. 4.7 and defined in equations (4.86) will be central to this development. We have already demonstrated that these operators play an important role in the case of monochromatic fields. For partially-coherent fields, this role is even more fundamental. We will start with statistical description of partially coherent fields and introduce the coherence matrix and the cross-spectral density. At this point, we will depart slightly from the convention and define the above quantities for real-valued physical fields rather than for analytic signals. This will allow us to avoid ambiguities and make the material more physically transparent. Then, some space will be devoted to the time-domain theory of the free-space Green's tensor and the T-matrix. Even though we will, eventually, express all quantities of interest in terms of the frequency-domain T-matrix and Π-operators, this material is needed to show that all derivations are mathematically consistent even though we work with stationary stochastic fields, which do not have temporal Fourier transforms. We will then state the general results for the extinguished, absorbed and scattered powers

due to an arbitrary partially-coherent incident field. The remainder of this section is devoted to special cases and approximations.

5.2 Statistical description of partially-coherent fields

The expression (4.91) used in Sec. 4.8 to represent a stationary non-monochromatic field is not of the most general form. Indeed, (4.91) defines an analytical, infinitely-differentiable function of time [12] at any point $\mathbf{r}$. Whether the actual physical fields possess this property is an open question. Indeed, Maxwell's equations only require existence of the first time derivative. Various models of radiation involve initial conditions that are applied instantaneously at random moments of time (e.g., excitation of an electronic state of a molecule by collision). Theoretically, fields produced by such sources are only piece-wise analytical and cannot be written in the form (4.91). Of course, instantaneous initial conditions is but a convenient approximation; more fundamental models can still predict infinitely-differentiable fields. Whether this reflects physical reality is hard to tell, and the relevant considerations touch on the origin of randomness in nature and the adequacy of PDE-based physical models.

Although all this can be viewed as fundamental for the theory of partially-coherent optical fields, we do not have adequate answers to the questions vaguely posed above and will sidestep them altogether. Instead, we will adopt a more phenomenological approach of the conventional theory of partial coherence [15]. In particular, we will not assume the exact knowledge of the incident field (or of any other field) as a function of time. Moreover, we will not require the fields to have temporal Fourier transforms or the associated analytic signals. In fact, contrary to the common physical intuition, it is not possible to give a mathematically-rigorous definition of Fourier transform for a general stationary stochastic process even with the use of generalized functions, finite time windows, etc. [13] Instead, we will characterize the fields by certain well-defined correlation functions, which are measurable and can be understood as time averages similar to the one introduced in (3.4). We will show that the absorbed, scattered and extinguished powers can be expressed in terms of such correlation functions.

We start by fixing some definitions. The *coherence matrix* of the field $\mathbf{E}(\mathbf{r}, t)$ is defined as the time average of the form

$$\Gamma(\mathbf{r}, \mathbf{r}'; \tau) = \langle \mathbf{E}(\mathbf{r}, t) \otimes \mathbf{E}(\mathbf{r}', t + \tau) \rangle . \tag{5.1}$$

---

[12] There exist various conceptually different definitions of continuity of a stochastic process. The process defined by (4.91) is said to be continuous in the sample path, but the regularity implied by (4.91) is even stronger: the sample path is not only continuous but infinitely differentiable for all (not just for most) realizations of the ensemble.

[13] A simple example of a bounded, zero mean, stationary stochastic process without a Fourier transform is the random telegraph signal [57]. A mathematically-consistent theory of power spectra of stationary stochastic processes was developed by Wiener [58], and a detailed discussion in a language that is more accessible to physicists was given by Wolf [11].

Unlike the temporal Fourier transform of $\mathbf{E}(\mathbf{r}, t)$, the average in (5.1) is well defined and can be computed directly from measured samples of any particular realization of the stochastic field. Note that $\mathbf{E}$ in the above definition is a real-valued measurable field and not an analytic signal. Consequently, Γ is a real 3×3 matrix. Here we depart from the conventional approach to define the coherence matrix in terms of the complex analytic signal. We do so because the analytic signal is neither a measurable quantity nor can be uniquely defined for a given realization of a stationary stochastic field, which we assume to physically exist, even though we can measure only a discrete finite set of its time samples. A detailed comparison of the definition adopted here to the conventional definition for the case of monochromatic fields is given in Sec. 7.2 below.

The coherence matrix satisfies the symmetry property

$$\Gamma(\mathbf{r}, \mathbf{r}'; -\tau) = \Gamma^{\mathsf{T}}(\mathbf{r}', \mathbf{r}; \tau) \, , \tag{5.2}$$

where the superscript T indicates matrix transposition. When $\mathbf{r}$ and $\mathbf{r}'$ are generic points, it is common to refer to $\Gamma(\mathbf{r}, \mathbf{r}'; \tau)$ as the *mutual* coherence matrix. In contrast, the special restriction $\Gamma(\mathbf{r}, \mathbf{r}; \tau)$ is frequently called the *autocorrelation* function or matrix. Below, we refer to Γ simply as the coherence matrix. It will be clear from the notations whether we consider two generic points $\mathbf{r}$ and $\mathbf{r}'$ or a single point $\mathbf{r}$. We can also introduce similar coherence matrices Γ_{inc} and Γ_{sca} for the incident and scattered field components in direct analogy to (5.1), which tacitly assumes that $\mathbf{E}$ is the total field. In fact, we will typically work with Γ_{inc} since the statistical properties of the incident field are assumed to be known or measurable and are therefore given as a condition of the problem.

The intensity of the field at the point $\mathbf{r}$ is defined as

$$I(\mathbf{r}) = \text{Tr}[\Gamma(\mathbf{r}, \mathbf{r}; 0)] \, . \tag{5.3}$$

Note that $I(\mathbf{r})$ can be related (up to an overall constant) to the current of electromagnetic energy only under special conditions, i.e., for propagating plane waves. It should be kept in mind that, in general, definition of the energy current (or density) involves both electric and magnetic fields whereas (5.3) involves the electric field only. Moreover, one can envisage a situation in which $I(\mathbf{r}) > 0$ at some point $\mathbf{r}$ but the current of energy (time-averaged Poynting vector) at this point is zero. Therefore, the word "intensity" should not be interpreted literally.

In most practical cases, the coherence matrix falls off with $|\tau|$ sufficiently fast so that its temporal Fourier transform exists [14]. The cross-spectral density matrix is defined as

$$\mathbb{W}(\mathbf{r}, \mathbf{r}'; \omega) = \int_{-\infty}^{\infty} \Gamma(\mathbf{r}, \mathbf{r}'; \tau) e^{\mathrm{i}\,\omega\tau} \mathrm{d}\tau \, , \tag{5.4a}$$

---

[14] In the case of fields with one or more strictly monochromatic components such as (4.91), the coherence matrix is oscillatory and does not fall off with τ. However, the temporal Fourier transform can still be defined in these cases with the use of generalized functions, see Sec. 7.2.

and the coherence matrix is related to the cross-spectral density by inverse Fourier transform,

$$\Gamma(\mathbf{r}, \mathbf{r}'; \tau) = \int\limits_{-\infty}^{\infty} \mathtt{W}(\mathbf{r}, \mathbf{r}'; \omega) e^{-\mathrm{i}\,\omega\tau} \frac{\mathrm{d}\omega}{2\pi} \ . \tag{5.4b}$$

Here we have again departed from the convention and defined $\mathtt{W}$ in (5.4a) without the factor of $1/2\pi$ in the right-hand side; this factor, of course, re-appears in the inverse transform (5.4b). This was done to preserve consistency with other Fourier definitions used in the chapter and elsewhere in physics. However, honoring the long-standing tradition, we give the cross-spectral density the special name $\mathtt{W}$ whereas elsewhere we use the same symbol for a function and its Fourier transform.

The cross-spectral density defined in (5.4) is a complex 3×3 matrix. It can be non-zero for both positive and negative frequencies. Its symmetry properties are derivable from those of Γ. From the definition (5.4), it is clear that the following element-wise property holds

$$\mathtt{W}(\mathbf{r}, \mathbf{r}'; -\omega) = \mathtt{W}^*(\mathbf{r}, \mathbf{r}'; \omega) \ . \tag{5.5a}$$

Additionally, we can utilize (5.2) to show that

$$\mathtt{W}(\mathbf{r}, \mathbf{r}'; -\omega) = \mathtt{W}^{\mathrm{T}}(\mathbf{r}', \mathbf{r}; \omega) \ . \tag{5.5b}$$

Taken together, (5.5a) and (5.5b) imply that $\mathtt{W}(\omega)$ has a Hermitian kernel:

$$\mathtt{W}(\mathbf{r}, \mathbf{r}'; \omega) = \mathtt{W}^{\dagger}(\mathbf{r}', \mathbf{r}; \omega) \ , \tag{5.5c}$$

where $\dagger$ indicates matrix transposition and complex conjugation of all elements (Hermitian conjugation). Note that the symmetry of the cross-spectral density is different from that of the T-matrix: the former is Hermitian while the latter is complex symmetric.

From the Hermiticity of $\mathtt{W}$, it follows that, for any sufficiently "nice" region $\mathbb{V}$ and for a fixed value of ω, we can expand $\mathtt{W}(\mathbf{r}, \mathbf{r}'; \omega)$ into a basis of orthonormal functions $\boldsymbol{F}_n(\mathbf{r}, \omega)$ as [11, 13]

$$\mathtt{W}(\mathbf{r}, \mathbf{r}'; \omega) = \sum_n w_n(\omega)\, \boldsymbol{F}_n(\mathbf{r}, \omega) \otimes \boldsymbol{F}_n^*(\mathbf{r}', \omega) \ , \quad \mathbf{r}, \mathbf{r}' \in \mathbb{V} \ , \tag{5.6a}$$

where

$$\int\limits_{\mathbb{V}} \mathtt{W}(\mathbf{r}, \mathbf{r}'; \omega) \boldsymbol{F}_n(\mathbf{r}', \omega)\, \mathrm{d}^3 r' = w_n(\omega) \boldsymbol{F}_n(\mathbf{r}, \omega) \ , \quad \mathbf{r} \in \mathbb{V} \ , \tag{5.6b}$$

$$\int\limits_{\mathbb{V}} \boldsymbol{F}_m^*(\mathbf{r}, \omega) \cdot \boldsymbol{F}_n(\mathbf{r}, \omega)\, \mathrm{d}^3 r = \delta_{nm} \ . \tag{5.6c}$$

Note that the eigenfunctions $\boldsymbol{F}_n(\mathbf{r}, \omega)$ can be complex. In operator notations, the expansion (5.6a) can be written as

$$\mathtt{W}(\omega) = \sum_n w_n(\omega) \, |\mathtt{F}_n(\omega)\rangle\langle\mathtt{F}_n(\omega)| \, , \quad \langle\mathtt{F}_n(\omega)|\mathtt{F}_m(\omega)\rangle = \delta_{nm} \, . \tag{5.6d}$$

It can be shown that the operator $\mathtt{W}(\omega)$ is non-negatively definite. Consequently,

$$w_n(\omega) \geq 0 \, . \tag{5.6e}$$

This property is proved in Sec. 7.3.

The significance of expansion (5.6b) will be explained below. For now, we note that, in some special cases, this expansion has only a few non-zero terms regardless of $\mathbb{V}$. In particular, if the incident field is a plane wave, then there are no more than two non-zero terms in the the expansion of the form (5.6a) for $\mathtt{W}_{\mathrm{inc}}(\omega)$. As a result, the extinction, absorption and scattering powers (or cross sections) are reduced in this case to simple spectral integrals. This is an instance of the intuitive spectral additivity that was discussed in Sec. 5.1 above. However, if (5.6b) has many non-negligible terms, the spectral additivity holds only in a generalized sense.

Some additional remarks about the expansion (5.6) can be made. First, in view of the symmetry properties (5.5), it is sufficient to know the expansion for positive frequencies only. In fact, all physical quantities can be expressed as integrals over positive frequencies. Secondly, it is important to keep in mind that the eigenfunctions $\boldsymbol{F}_n(\mathbf{r}, \omega)$ and the eigenvalues $w_n(\omega)$ depend not only on the statistical properties of the field but also on the shape of the region over which the integral operator is defined. While we can use to this end any sufficiently nice region of space, the energy-related quantities discussed in this chapter involve integrals over the region occupied by the target, $\mathbb{V}$. This is reflected in the equations (5.6).

The power spectrum (or simply the spectrum) of the field at a point $\mathbf{r}$ is defined as

$$S(\mathbf{r}, \omega) = \mathrm{Tr}[\mathtt{W}(\mathbf{r}, \mathbf{r}; \omega)] \, . \tag{5.7}$$

Here, due to another long-standing tradition, we have used the same symbol S for the power spectrum as for the Poynting vector. However, it should always be clear which quantity we mean from the context and the font used: the Poynting vector is always denoted by a bold letter $\mathbf{S}$ whereas the spectrum is a scalar and denoted by the plain Italic letter S.

As follows from (5.5c), the diagonal elements $W_{ii}(\mathbf{r}, \mathbf{r}; \omega)$ are real-valued. Moreover, each diagonal element $W_{ii}(\mathbf{r}, \mathbf{r}; \omega)$ is a power spectrum of some scalar stochastic process; this quantity is known to be non-negative (see Sec. 7.3 for a proof). Therefore, $S(\mathbf{r}, \omega)$ is also non-negative. Note that

$$\int_{\mathbb{V}} S(\mathbf{r}, \omega) \mathrm{d}^3 r = \sum_n w_n(\omega) \, , \tag{5.8}$$

where $w_n(\omega)$ are the eigenvalues of $\mathbb{W}(\omega)$.

An additional quantity related to the coherence matrix is the degree of coherence. It is defined in Sec. 7.2 below but is not used directly to compute any power-related quantities.

5.3 Time-domain Green's tensor and T-matrix

In Sec. 4, we have introduced the frequency-domain, free space Greens tensor G and the T-matrix of the target, T. The Green's tensor gives the scattered field $\mathbf{E}_{\mathrm{sca}}$ produced by an arbitrary monochromatic polarization $\mathbf{P}$ according to (4.9) and the T-matrix gives $\mathbf{P}$ in terms of an arbitrary incident field $\mathbf{E}_{\mathrm{inc}}$ according to (4.49). These equations couple the complex amplitudes or, equivalently, the Fourier components of the fields in question at a given oscillation frequency ω. However, if the incident field is stochastic and stationary, temporal Fourier transforms of $\mathbf{E}_{\mathrm{inc}}$, $\mathbf{P}$ and $\mathbf{E}_{\mathrm{sca}}$ do not exist. In this case, equations (4.9) and (4.49) are inapplicable and we must resort to the time-domain equations. Naturally, these equations involve the time-domain Green's tensor and T-matrix.

Although the time- and frequency-domain G and T are related to each other by the temporal Fourier transform, i.e.,

$$\mathsf{G}(\mathbf{r},\mathbf{r}';\omega) = \int_{-\infty}^{\infty} \mathsf{G}(\mathbf{r},\mathbf{r}';\tau)e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau \ , \quad \mathsf{G}(\mathbf{r},\mathbf{r}';\tau) = \int_{-\infty}^{\infty} \mathsf{G}(\mathbf{r},\mathbf{r}';\omega)e^{-\mathrm{i}\,\omega\tau}\frac{\mathrm{d}\omega}{2\pi} \ , \quad (5.9\mathrm{a})$$

$$\mathsf{T}(\mathbf{r},\mathbf{r}';\omega) = \int_{-\infty}^{\infty} \mathsf{T}(\mathbf{r},\mathbf{r}';\tau)e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau \ , \quad \mathsf{T}(\mathbf{r},\mathbf{r}';\tau) = \int_{-\infty}^{\infty} \mathsf{T}(\mathbf{r},\mathbf{r}';\omega)e^{-\mathrm{i}\,\omega\tau}\frac{\mathrm{d}\omega}{2\pi} \ , \quad (5.9\mathrm{b})$$

the time-domain operators are more fundamental: they are applicable to all physically-realizable fields. We therefore can use the time-domain Green's tensor and T-matrix without restriction. If the fields are Fourier-transformable, then we can convert the time-domain equations into frequency domain and work with each Fourier component separately. But making this transition is not always possible, at least, not using a naive approach, which assumes that all fields are Fourier-transformable [15]. Therefore, to derive all results in a mathematically consistent way, we will start from the time-domain equations. In this subsection, we develop the relevant mathematical formalism.

The time-domain counter-parts of equations (4.9) and (4.49) are

$$\mathbf{E}_{\mathrm{sca}}(\mathbf{r},t) = \int_{\mathbb{V}} \mathrm{d}^3 r' \int_{0}^{\infty} \mathrm{d}\tau \, \mathsf{G}(\mathbf{r},\mathbf{r}';\tau)\mathbf{P}(\mathbf{r}',t-\tau) \ , \qquad (5.10\mathrm{a})$$

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[15] Another important example where the frequency-domain equations should be used with caution are amplifying media [59, 60].

$$\mathbf{P}(\mathbf{r}, t) = \int_{\mathbb{V}} \mathrm{d}^3 r' \int_0^{\infty} \mathrm{d}\tau \, \mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau) \mathbf{E}_{\mathrm{inc}}(\mathbf{r}', t - \tau) \, . \qquad (5.10b)$$

While we state below a general closed-form expression for the Green's tensor $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \tau)$, the T-matrix $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau)$ depends on the target properties in a complicated way. We can only write an integral equation, which, in principle, defines $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau)$. Solving this equation is a problem of numerical analysis.

The time-domain Green's tensor (also known as the retarded Green's tensor) can be written concisely in the form [61, $72]

$$\mathsf{G}(\mathbf{r}, \mathbf{r}'; \tau) = (\nabla \times \nabla \times) \frac{1}{|\mathbf{r} - \mathbf{r}'|} \delta\left(\tau - |\mathbf{r} - \mathbf{r}'|/c\right) \, . \qquad (5.11)$$

Here the spatial derivatives are evaluated with respect to the radius-vector of the point of observation, $\mathbf{r}$. It can be easily verified that the Fourier transform of (5.11) into the frequency domain yields (4.10). The expression (5.11) is, however, not very convenient in practical computations. We can obtain a more manageable formula either by computing the derivatives in (5.11) directly (in this case, we should be careful not to lose the singular part) or by utilizing the Fourier transform (5.9a) and the already derived frequency-domain expression (4.10) for $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega)$. However, if we follow the latter route, it would be more convenient to start from the decomposition (4.10a) of $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \omega)$ into the singular and regular parts and then use the decomposition (4.13) for the regular part. A simple calculation performed along these lines yields

$$\mathsf{G}(\mathbf{r}, \mathbf{r}'; \tau) = -\frac{4\pi}{3} \delta(\mathbf{r} - \mathbf{r}') \delta(\tau) \mathsf{I}_3 \, + \, \mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \tau) \, , \qquad (5.12a)$$

$$\mathsf{G}_{\mathrm{R}}(\mathbf{r}, \mathbf{r}'; \tau) = \left[\mathsf{G}_0(\mathbf{r}, \mathbf{r}') - \mathsf{G}_1(\mathbf{r}, \mathbf{r}') \frac{\partial}{c \partial t} + \mathsf{G}_2(\mathbf{r}, \mathbf{r}') \frac{\partial^2}{c^2 \partial \tau^2} \right] \delta(\tau - |\mathbf{r} - \mathbf{r}'|/c) \, .$$
$$(5.12b)$$

The time- and frequency-independent tensors G_0, G_1 and G_2 are defined in (4.13b) through (4.13d). Time derivatives of the delta-function should be interpreted in the sense of integration by parts, i.e.,

$$\int f(\tau) [\partial \delta(\tau - \tau_0) / \partial \tau] \mathrm{d}\tau = - \left. \partial f(\tau) / \partial \tau \right|_{\tau = \tau_0} \, . \qquad (5.13)$$

Thus, similarly to the frequency-domain case, we have isolated the singular part of $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \tau)$; the regular part defined in (5.12b) integrates to zero over a small ball $|\mathbf{r} - \mathbf{r}'| \le a \to 0$ independently of time. This decomposition is useful for constructing integral equations and perturbation expansions. Note that $\mathsf{G}(\mathbf{r}, \mathbf{r}'; \tau)$ is zero for $\tau < 0$. Also, we can evaluate the delta-functions and write the scattered field in terms of polarization and its time derivatives as

$$\mathbf{E}_{\text{sca}}(\mathbf{r}, t) = -\frac{4\pi}{3}\mathbf{P}(\mathbf{r}, t) \ +$$

$$\int_{\mathbb{V}} \mathrm{d}^3 r' \left[\mathsf{G}_0(\mathbf{r}, \mathbf{r}') + \mathsf{G}_1(\mathbf{r}, \mathbf{r}')\frac{\partial}{c\partial t} + \mathsf{G}_2(\mathbf{r}, \mathbf{r}')\frac{\partial^2}{c^2\partial t^2} \right] \mathbf{P}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c) . \quad (5.14)$$

The first term in (5.14) (due to the singularity of Green's tensor) is zero in free space.

The time-domain integral equation for polarization is of the form

$$\mathbf{P}(\mathbf{r}, t) = \int_0^\infty \mathrm{d}\tau \, \kappa(\mathbf{r}, \tau)\mathbf{E}_{\text{inc}}(\mathbf{r}, \ t - \tau)$$

$$+ \int_0^\infty \mathrm{d}\tau \int_0^\infty \mathrm{d}\eta \int_{\mathbb{V}} \mathrm{d}^3 r' \, \kappa(\mathbf{r}, \tau)\mathsf{G}_{\text{R}}(\mathbf{r}, \mathbf{r}'; \eta)\mathbf{P}(\mathbf{r}', \ t - \tau - \eta) . \qquad (5.15)$$

The coupling function $\kappa(\mathbf{r}, \tau)$ is the Fourier transform of the frequency-domain coupling function $\kappa(\mathbf{r}, \omega)$, which was defined in (4.18). We have

$$\kappa(\mathbf{r}, \tau) = \int_{-\infty}^\infty \frac{\chi(\mathbf{r}, \omega) \, e^{-\mathrm{i}\,\omega\tau}}{1 + \frac{4\pi}{3}\chi(\mathbf{r}, \omega)} \frac{\mathrm{d}\omega}{2\pi} = \frac{3}{4\pi} \int_{-\infty}^\infty \frac{\epsilon(\mathbf{r}, \omega) - 1}{\epsilon(\mathbf{r}, \omega) + 2} \, e^{-\mathrm{i}\,\omega\tau} \, \frac{\mathrm{d}\omega}{2\pi} . \qquad (5.16)$$

While it is easy to express $\kappa(\mathbf{r}, \omega)$ in terms of $\chi(\mathbf{r}, \omega)$ or $\epsilon(\mathbf{r}, \omega)$, a similar algebraic relation cannot be given in time domain. Rather, the expression for $\kappa(\mathbf{r}, \tau)$ in terms of $\chi(\mathbf{r}, \tau)$ involves a quadrature. However, in many special cases, the integral can be computed analytically. This includes the case of media with Lorentz dispersion, see Sec. 7.1. Note that $\kappa(\mathbf{r}, \tau)$ can be defined on the whole real axis of τ. Just like the susceptibility $\chi(\mathbf{r}, \tau)$, the coupling function $\kappa(\mathbf{r}, \tau)$ is causal, which means that it is zero for $\tau < 0$.

We can further utilize the expression (5.12b) to simplify the double time integral in (5.15). Using integration by parts and the asymptotic property $\kappa(\mathbf{r}, \tau) \to 0$ as $|\tau| \to \infty$, (5.15) can be transformed into

$$\mathbf{P}(\mathbf{r}, t) = \int_0^\infty \mathrm{d}\tau \, \varkappa_0(\mathbf{r}, \tau)\mathbf{E}_{\text{inc}}(\mathbf{r}, t - \tau) + \int_{\mathbb{V}} \mathrm{d}^3 r' \int_0^\infty \mathrm{d}\tau \left[\varkappa_0(\mathbf{r}, \tau)\mathsf{G}_0(\mathbf{r}, \mathbf{r}') \right.$$

$$\left. + \varkappa_1(\mathbf{r}, \tau)\mathsf{G}_1(\mathbf{r}, \mathbf{r}') + \varkappa_2(\mathbf{r}, \tau)\mathsf{G}_2(\mathbf{r}, \mathbf{r}') \right]\mathbf{P}(\mathbf{r}', \ t - \tau - |\mathbf{r} - \mathbf{r}'|/c) , \quad (5.17)$$

where

$$\varkappa_n(\mathbf{r}, \tau) = \frac{1}{c^n} \frac{\partial^n \kappa(\mathbf{r}, \tau)}{\partial \tau^n} , \quad n = 0, 1, 2 . \qquad (5.18)$$

Thus, in the most general case, the integral equation for polarization contains no more than one time integral.

We can eliminate time integration in (5.18) altogether if the target is non-dispersive in the frequency range of interest, i.e., in the case when the spectrum of the incident field is supported in the transparency window of the target material where $\epsilon(\omega)$ is approximately real and constant. Assume for simplicity that the target is also spatially uniform. These approximations often hold with good precision in scattering problems. Then the coupling function can be approximated as $\kappa(\mathbf{r}, \tau) = \kappa_0 \delta(\tau)$ for $\mathbf{r} \in \mathbb{V}$ where κ_0 is a positive constant. In this case, the integral equation (5.15) takes the form

$$
\begin{aligned}
\mathbf{P}(\mathbf{r}, t) = \kappa_0 \Bigg\{ \mathbf{E}_{\text{inc}}(\mathbf{r}, t) + \int_{\mathbb{V}} \mathrm{d}^3 r' \ \Bigg[\mathsf{G}_0(\mathbf{r}, \mathbf{r}') + \ \mathsf{G}_1(\mathbf{r}, \mathbf{r}') \frac{\partial}{c \partial t} \\
+ \ \mathsf{G}_2(\mathbf{r}, \mathbf{r}') \frac{\partial^2}{c^2 \partial t^2} \Bigg] \ \mathbf{P}(\mathbf{r}', \ t - |\mathbf{r} - \mathbf{r}'|/c) \Bigg\} \quad \text{(non-dispersive target)} .
\end{aligned}
$$
(5.19)

We are thus left with two time derivatives but no time integration.

We now turn to the T-matrix. The latter is, essentially, the linear operator yielding the solution to the integral equation (5.15) in terms of the incident field. Substituting the ansatz (5.10b) into (5.15), we obtain the equation

$$
\begin{aligned}
\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau) = \kappa(\mathbf{r}, \tau) \delta(\mathbf{r} - \mathbf{r}') \mathsf{I}_3 \\
+ \int_{\mathbb{V}} \mathrm{d}^3 r'' \int_0^\infty \mathrm{d}\eta \int_0^\infty \mathrm{d}\zeta \ \kappa(\mathbf{r}, \eta) \mathsf{G}_{\mathsf{R}}(\mathbf{r}, \mathbf{r}''; \zeta) \mathsf{T}(\mathbf{r}'', \mathbf{r}'; \tau - \eta - \zeta) .
\end{aligned}
$$
(5.20)

Note that it follows from the causality of $\kappa(\mathbf{r}, \tau)$ that $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau)$ is also causal (is zero for $\tau < 0$). Equation (5.20) is rather complicated in general but it can be simplified under various assumptions. Assuming, as above, that the target is non-dispersive and spatially-uniform so that $\kappa(\mathbf{r}, \tau) = \kappa_0 \delta(\tau)$ for $\mathbf{r} \in \mathbb{V}$, we obtain the simplified equation

$$
\begin{aligned}
\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau) = \kappa_0 \Bigg\{ \delta(\mathbf{r} - \mathbf{r}') \delta(\tau) \mathsf{I}_3 + \int_{\mathbb{V}} \mathrm{d}^3 r'' \ \Bigg[\mathsf{G}_0(\mathbf{r}, \mathbf{r}'') + \mathsf{G}_1(\mathbf{r}, \mathbf{r}'') \frac{\partial}{c \partial \tau} \\
+ \mathsf{G}_2(\mathbf{r}, \mathbf{r}'') \frac{\partial^2}{c^2 \partial \tau^2} \Bigg] \ \mathsf{T}(\mathbf{r}'', \mathbf{r}'; \ \tau - |\mathbf{r} - \mathbf{r}'|/c) \Bigg\} .
\end{aligned}
$$
(5.21)

This equation is, in principle, amenable to numerical simulations. We emphasize that inversion of large matrices is not required to solve a discretized version of (5.21). Rather, we can update $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau_k)$ at each new value of discretized time $\tau_k = (\delta\tau)k$ by evaluating the right-hand side directly. This approach however requires keeping in memory sufficiently long histories of $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau_l)$ with $l < k$. Appropriate initial conditions are also required.

5.4 General results

We now derive the results for extinguished, absorbed and scattered powers for incident fields of arbitrary form and state of coherence. The expressions obtained below are amenable to numerical computations. One will need to know the T-matrix $\mathtt{T}(\omega)$ or the scattering operator $\Sigma(\omega)$ of the target as defined in Sec. 4.7 and the cross-spectral density of the incident field $\mathtt{W}_{\text{inc}}(\omega)$ over a sufficiently wide frequency band. The scattering operator (and, by extension, the T-matrix) can be computed by the DDA or by any other method for solving volume integral equations [16]. Regarding $\mathtt{W}_{\text{inc}}(\omega)$, there exist various theoretical models for the cross-spectral density [55,56], and, in principle, it can also be measured experimentally.

5.4.1 Extinguished, absorbed and scattered powers in terms of the incident field

We start with extinction; results for absorption and scattering will be obtained by direct analogy. The general expression for extinguished power under illumination by a stationary, partially-coherent incident field can be obtained from the local representation (3.19) and the constitutive relation (3.5). Combining the two equations, we obtain

$$Q_{\text{ext}} = \int_{\mathbb{V}} \left\langle \frac{\partial \mathbf{P}(\mathbf{r}, t)}{\partial t} \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \right\rangle \mathrm{d}^3 r \ . \tag{5.22}$$

Substituting into this expression $\mathbf{P}(\mathbf{r}, t)$ from (5.10b), we find that

$$Q_{\text{ext}} = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \, q_{\text{ext}}(\mathbf{r}, \mathbf{r}') \ , \tag{5.23}$$

where

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = \left\langle \int_0^\infty \left[\frac{\partial}{\partial t} \mathtt{T}(\mathbf{r}, \mathbf{r}'; \tau) \mathbf{E}_{\text{inc}}(\mathbf{r}', t - \tau) \right] \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \, \mathrm{d}\tau \right\rangle \ . \tag{5.24}$$

Here, in order to shorten the formulas, we have introduced the density of extinction $q_{\text{ext}}(\mathbf{r}, \mathbf{r}')$. The scope of differentiation with respect to t in (5.24) is restricted to the expression in the square brackets. Using the identity $\partial f(t - \tau)/\partial t = -\partial f(t - \tau)/\partial \tau$, we can re-write (5.24) as

---

[16] PDE-based solvers such as the finite difference or finite element methods can also be used, but are not as convenient in this context since these methods do not account analytically for the boundary conditions at infinity and may require modification of the physical model by introducing perfectly-matched layers (PMLs) and similar artificial constructs.

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = -\int\limits_0^\infty \left\langle \mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau) \frac{\partial}{\partial \tau} \mathbf{E}_{\text{inc}}(\mathbf{r}', t - \tau) \cdot \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \right\rangle \, d\tau \; . \qquad (5.25)$$

Note that the T-matrix in the above expression is independent of the averaging variable, t. Recalling the definition of the coherence matrix (5.1), we obtain the following expression for the density of extinction:

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = -\text{Tr} \int\limits_0^\infty \mathsf{T}(\mathbf{r}, \mathbf{r}'; \tau) \frac{\partial}{\partial \tau} \Gamma_{\text{inc}}(\mathbf{r}', \mathbf{r}; \tau) \, d\tau \; . \qquad (5.26)$$

Here Γ_{inc} is the coherence matrix of the incident field.

Equation (5.26) is the most general expression for the density of extinction. It depends on the properties of the target (encoded mathematically in its T-matrix) and on the statistical properties of the incident field (encoded in the coherence matrix). Notably, all functions of τ in (5.26) have well-defined Fourier transforms. We can therefore re-write (5.26) in terms of the frequency-domain quantities. The easiest way to accomplish this is to take into account causality of the T-matrix (it is zero for $\tau < 0$) and expand integration over τ in (5.26) to the whole real axis; then substitute the Fourier representations (5.4b) and (5.9b) for Γ and T into the integrand and compute the derivative $\partial/\partial\tau$. This results in the following expression:

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = \mathrm{i} \, \text{Tr} \int\limits_{-\infty}^\infty \frac{\omega d\omega}{2\pi} \int\limits_{-\infty}^\infty \frac{d\omega'}{2\pi} \int\limits_{-\infty}^\infty d\tau \, e^{-\mathrm{i}(\omega + \omega')\tau} \, \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega') \, \mathsf{W}_{\text{inc}}(\mathbf{r}', \mathbf{r}; \omega) \; .$$
$$(5.27)$$

We can now compute the integral over τ using the Fourier representation of the delta-function

$$\int\limits_{-\infty}^\infty e^{-\mathrm{i}(\omega + \omega')\tau} \, d\tau = 2\pi \, \delta(\omega + \omega') \; . \qquad (5.28)$$

Substituting this back into (5.27), we obtain

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = -\frac{\mathrm{i}}{2\pi} \, \text{Tr} \int\limits_{-\infty}^\infty \mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega) \, \mathsf{W}_{\text{inc}}(\mathbf{r}', \mathbf{r}; -\omega) \, \omega d\omega \; . \qquad (5.29)$$

The same result can be obtained without extending integration in (5.26) to the whole real axis, but the derivation in this case is more tedious and based on the analytical properties of $\mathsf{W}_{\text{inc}}(\mathbf{r}', \mathbf{r}; \omega)$. Next, we use the symmetry properties (4.58a) and (5.5a) to transform (5.29) to an integral over positive frequencies only. We thus arrive at the expression

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} \, \text{Im} \, \text{Tr} \int_0^{\infty} \text{T}(\mathbf{r}, \mathbf{r}'; \omega) \, \text{W}_{\text{inc}}(\mathbf{r}', \mathbf{r}; -\omega) \, \omega \mathrm{d}\omega \, . \qquad (5.30)$$

We can further re-write this equation using the additional (less trivial) symmetry properties of T and W (4.58b) and (5.5b) and the matrix identity $\text{Tr}(\text{A}^{\text{T}}\text{B}^{\text{T}}) = \text{Tr}(\text{BA})$

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} \, \text{Im} \, \text{Tr} \int_0^{\infty} \text{T}(\mathbf{r}, \mathbf{r}'; \omega) \, \text{W}_{\text{inc}}^{\text{T}}(\mathbf{r}, \mathbf{r}'; \omega) \, \omega \mathrm{d}\omega \qquad (5.31a)$$

$$= \frac{1}{\pi} \, \text{Im} \, \text{Tr} \int_0^{\infty} \text{W}_{\text{inc}}(\mathbf{r}, \mathbf{r}'; \omega) \, \text{T}(\mathbf{r}', \mathbf{r}; \omega) \, \omega \mathrm{d}\omega \, . \qquad (5.31b)$$

The second expression is particularly simple. Both expressions can be viewed as statements of generalized spectral additivity of extinction. Indeed, these formulas contain an integral over positive frequencies of a product of two functions. The first function is a quadratic combination of the incident field, which characterizes in the most general way its spectrum. The second function characterizes linear response of the target in frequency domain. Note that (5.31) gives the density of extinction, which is a non-local quantity. To obtain the total extinguished power, we must integrate the density according to (5.23). However, in many special cases including the case of an incident plane wave (Sec. 5.8) and the quasi-static approximation (Sec. 5.9) this complicated integration can be carried out analytically and the result for Q_{ext} is significantly simplified.

The spatial integrals in (5.23) can be computed analytically if we know the representation of the operators $\text{W}(\omega)$ and $\text{T}(\omega)$ in the same orthonormal basis. A convenient basis is formed by the eigenfunctions $\boldsymbol{F}_n(\mathbf{r}, \omega)$, which appear in the expansion of $\text{W}(\omega)$ (5.6a). Substituting this expansion into (5.31b), we obtain for the density of extinction

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi} \, \text{Im} \int_0^{\infty} \omega \mathrm{d}\omega \sum_n w_n(\omega) \boldsymbol{F}_n^*(\mathbf{r}', \omega) \cdot \text{T}(\mathbf{r}', \mathbf{r}; \omega) \boldsymbol{F}_n(\mathbf{r}, \omega) \, . \qquad (5.32)$$

Recalling that the total extinguished power is an integral of $q_{\text{ext}}(\mathbf{r}, \mathbf{r}')$, we obtain

$$Q_{\text{ext}} = \frac{1}{\pi} \int_0^{\infty} \omega \mathrm{d}\omega \sum_n w_n(\omega) \, \text{Im} \, \langle \text{F}_n(\omega)|\text{T}(\omega)|\text{F}_n(\omega) \rangle \, , \qquad (5.33)$$

where

$$\langle \text{F}_n(\omega)|\text{T}(\omega)|\text{F}_n(\omega) \rangle = \int_{\mathbb{V}} \mathrm{d}^3 r_1 \int_{\mathbb{V}} \mathrm{d}^3 r_2 \, \boldsymbol{F}_n^*(\mathbf{r}_1, \omega) \cdot \text{T}(\mathbf{r}_1, \mathbf{r}_2; \omega) \boldsymbol{F}_n(\mathbf{r}_2, \omega) \, .$$

$$(5.34)$$

The result (5.33) is remarkable because the expression

$$\mathcal{Q}_{\text{ext}}^{(n)}(\omega) = \frac{\omega}{2}\,\text{Im}\,\langle \mathsf{F}_n(\omega)|\mathsf{T}(\omega)|\mathsf{F}_n(\omega)\rangle \tag{5.35}$$

formally coincides with the extinguished power for an incident monochromatic field of the form $\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \text{Re}[\boldsymbol{F}_n(\mathbf{r}, \omega)e^{-\mathrm{i}\,\omega t}]$. This can be seen by comparing (5.33) to (4.85) and (4.86). We have used the curly font in $\mathcal{Q}_{\text{ext}}^{(n)}(\omega)$ because $\boldsymbol{F}_n(\mathbf{r}, \omega)$, as defined, does not have the units of electric field; correspondingly, $\mathcal{Q}_{\text{ext}}^{(n)}(\omega)$ does not have the units of power. In fact, the units of $\mathcal{Q}_{\text{ext}}^{(n)}(\omega)$ are those of frequency. The total extinguished power is given by

$$Q_{\text{ext}} = \frac{2}{\pi}\sum_n \int_0^\infty w_n(\omega)\mathcal{Q}_{\text{ext}}^{(n)}(\omega)\mathrm{d}\omega \;. \tag{5.36}$$

The overall factor $2/\pi$ originates from the definitions of Fourier transform and the way complex amplitudes of monochromatic fields are related to the real-valued quantities. Focusing on a fixed frequency, we can say that $(2/\pi)w_n(\omega)$ quantifies how much energy (per unit frequency) the incident field has in the n-th mode whereas $\mathcal{Q}_{\text{ext}}^{(n)}(\omega)$ quantifies how much of this energy is extinguished by the target.

The above results illustrate physical significance of the expansion (5.6b). We see that every eigenfunction $\boldsymbol{F}_n(\mathbf{r}, \omega)$ is equivalent (up to a constant factor of the dimensionality of electric field divided by square root of volume) to the complex amplitude of some strictly monochromatic field oscillating at the frequency ω. This field is extinguished by the target independently of any other mode at any other frequency, and the total extinguished power is a sum over ω and n. We can say that the two modes $\boldsymbol{F}_n(\mathbf{r}, \omega)$ and $\boldsymbol{F}_m(\mathbf{r}, \omega')$ are mutually incoherent and do not exhibit any interference effects in extinction if the pairs (n, ω) and (m, ω') are distinct. Of course, there could be several or many modes $\boldsymbol{F}_n(\mathbf{r}, \omega)$ with non-negligible weights $w_n(\omega)$ at any given frequency. This is so because, in general, a partially-coherent field is not a superposition of monochromatic fields oscillating at different frequencies; in other words, a partially-coherent field is not a Fourier transform of some function of frequency.

Similar results for the absorbed and scattered powers can be obtained by replacing $\Pi_{\text{ext}}(\omega) = \mathsf{T}(\omega)$ with the appropriate Π-operator or H-operator defined in (4.86) and (4.88). The general result for the density (power per unit volume squared) can be stated as

$$q_*(\mathbf{r}, \mathbf{r}') = \frac{1}{\pi}\,\text{Im Tr}\int_0^\infty \mathbb{W}_{\text{inc}}(\mathbf{r}, \mathbf{r}'; \omega)\,\Pi_*(\mathbf{r}', \mathbf{r}; \omega)\,\omega\mathrm{d}\omega \tag{5.37a}$$

$$= \frac{1}{\pi}\,\text{Re Tr}\int_0^\infty \mathbb{W}_{\text{inc}}(\mathbf{r}, \mathbf{r}'; \omega)\,\mathsf{H}_*(\mathbf{r}', \mathbf{r}; \omega)\,\omega\mathrm{d}\omega \;, \tag{5.37b}$$

where $*$ stands for either abs, sca or ext. Note that the operator of real part in (5.37b) can be omitted. As both $\mathtt{W}(\omega)$ and $\mathtt{H}_*(\omega)$ are Hermitian, imaginary part of the right-hand side of (5.37b) integrates to zero when used in (5.23). However, without the real part, the formula (5.37b) would define a complex density of extinction, which we wish to avoid.

For the total powers, we have

$$Q_* = \frac{1}{\pi} \operatorname{Im} \int_0^\infty \omega \mathrm{d}\omega \sum_n w_n(\omega) \langle \mathrm{F}_n(\omega) | \Pi_*(\omega) | \mathrm{F}_n(\omega) \rangle \qquad (5.38\mathrm{a})$$

$$= \frac{2}{\pi} \sum_n \int_0^\infty w_n(\omega) \mathcal{Q}_*^{(n)}(\omega) \mathrm{d}\omega . \qquad (5.38\mathrm{b})$$

where

$$\mathcal{Q}_*^{(n)}(\omega) = \frac{\omega}{2} \operatorname{Im} \langle \mathrm{F}_n(\omega) | \Pi_*(\omega) | \mathrm{F}_n(\omega) \rangle . \qquad (5.39)$$

5.4.2 Extinction in terms of imaginary part of the T-matrix

Here we adduce one more general expression for Q_{ext} in which the imaginary part of the T-matrix appears explicitly but its real part is not present. Following the same approach as in Sec. 4.6, we can account for the symmetries of $\mathtt{W}_{\mathrm{inc}}$ and $\mathtt{T}$ as follows. Let us write

$$\mathtt{W}_{\mathrm{inc}}(\mathbf{r}, \mathbf{r}'; \omega) = \mathtt{W}_{\mathrm{inc}}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}'; \omega) + \mathrm{i}\ \mathtt{W}_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega) , \qquad (5.40)$$

where $\mathtt{W}_{\mathrm{inc}}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}'; \omega)$ and $\mathtt{W}_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega)$ are purely real. Note that the first kernel is symmetric and the second is anti-symmetric:

$$[\mathtt{W}_{\mathrm{inc}}^{(\mathrm{r})}(\mathbf{r}, \mathbf{r}'; \omega)]_{ij} = \ [\mathtt{W}_{\mathrm{inc}}^{(\mathrm{r})}(\mathbf{r}', \mathbf{r}; \omega)]_{ji} , \qquad (5.41\mathrm{a})$$

$$[\mathtt{W}_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega)]_{ij} = -[\mathtt{W}_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}', \mathbf{r}; \omega)]_{ji} . \qquad (5.41\mathrm{b})$$

This follows from the Hermiticity of $\mathtt{W}(\omega)$ as expressed in (5.5c). On the other hand, the T-matrix is symmetric:

$$[\mathtt{T}_{\mathrm{inc}}(\mathbf{r}, \mathbf{r}'; \omega)]_{ij} = [\mathtt{T}_{\mathrm{inc}}(\mathbf{r}', \mathbf{r}; \omega)]_{ji} . \qquad (5.41\mathrm{c})$$

It can be seen that

$$\operatorname{Tr} \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \ \mathtt{W}_{\mathrm{inc}}^{(\mathrm{i})}(\mathbf{r}, \mathbf{r}'; \omega) \ \mathtt{T}(\mathbf{r}', \mathbf{r}; \omega) = 0 . \qquad (5.42)$$

Consequently,

$$Q_{\text{ext}} = \frac{1}{\pi} \text{Tr} \int\limits_{\mathbb{V}} \mathrm{d}^3 r \int\limits_{\mathbb{V}} \mathrm{d}^3 r' \int\limits_0^\infty \mathtt{W}_{\text{inc}}^{(\mathbf{r})}(\mathbf{r}, \mathbf{r}'; \omega) \, \mathtt{T}^{(\mathtt{i})}(\mathbf{r}', \mathbf{r}; \omega) \, \omega \mathrm{d}\omega \,. \qquad (5.43)$$

This expression is similar to (4.87), (4.88c) and, in the case of illumination by a monochromatic field, is reduced to the latter.

Thus, what matters for extinction is the real part of the cross-spectral density and imaginary part of the T-matrix. While the densities (5.31) and (5.43) yield the same Q_{ext} when used in the double integral (5.23), the form (5.31) is often more convenient or easier to handle mathematically. However, we will find the form (5.43) useful in Sec. 5.7 below where we cover quasi-monochromatic field.

5.4.3 Absorption in terms of the total field

Above, we have expressed the absorbed, scattered and extinguished powers in terms of statistical properties of the incident field. In the case of absorption, we can also obtain an expression in terms of the total field. Such a result may be less practically-relevant as, typically, properties of the incident field are given as conditions of the problem. However, there exist numerical solvers of Maxwell's equations that compute the total field rather than the T-matrix or the scattering operator. If statistical properties of the total field are known, a very simple expression for the absorbed power can be written.

We start with the local representation of the absorbed power (3.16) and, using the constitutive relation (3.5), obtain after a few simple manipulations the expression

$$Q_{\text{abs}} = -\text{Tr} \int\limits_{\mathbb{V}} \mathrm{d}^3 r \int\limits_0^\infty \mathrm{d}\tau \, \chi(\mathbf{r}, \tau) \frac{\partial}{\partial \tau} \Gamma(\mathbf{r}, \mathbf{r}; \tau) \,. \qquad (5.44)$$

Note that Γ in the above formula is the coherence matrix for the total field. We can convert this expression to frequency domain using essentially the same steps as in Sec. 5.4.1 above. The resulting expression is

$$Q_{\text{abs}} = \frac{1}{\pi} \text{Im} \int\limits_{\mathbb{V}} \mathrm{d}^3 r \int\limits_0^\infty \omega \mathrm{d}\omega \, \chi(\mathbf{r}, \omega) S(\mathbf{r}, \omega) \,, \qquad (5.45)$$

where $S(\mathbf{r}, \omega)$ is the power spectrum of the total field defined in (5.7). Thus, to apply (5.45), we need a numerical solver, which computes the power spectrum of the total field given statistical properties of the incident field and the properties of the target. This is not what most existing computational packages do, but is possible in principle because the coherence matrix satisfies, essentially, the same equations as the field.

The expression (5.45) is one of the many faces of the fluctuation-dissipation theorem [62]. It looks much simpler than the corresponding expression in terms of

the incident field (5.38). However, the complexity of any practical computation is hidden in the task of computing $S(\mathbf{r}, \omega)$.

We can show for consistency that (5.45) agrees with the previously-derived result for monochromatic fields. Indeed, for the total electric field of the form (4.2) oscillating at the frequency ω_0, we have $S(\mathbf{r}, \omega) = (\pi/2)|\boldsymbol{E}(\mathbf{r})|^2[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$ (see (7.27) in Sec. 7.2). We therefore obtain

$$Q_{\text{abs}} = \frac{\omega}{8\pi} \int_{\mathbb{V}} \text{Im}[\epsilon(\mathbf{r}, \omega_0)] \, |\boldsymbol{E}(\mathbf{r})|^2 \text{d}^3 r \quad \text{(for monochromatic field)} , \qquad (5.46)$$

where we have used $\text{Im}[\chi(\mathbf{r}, \omega)] = \text{Im}[\epsilon(\mathbf{r}, \omega)]/4\pi$. As expected, this expression is identical to the previously-derived formula (4.7a).

5.5 Independent sources

Consider the case when the incident field is produced by two physically-independent sources. By this we mean that each source emits radiation independently of existence or nonexistence or properties of other sources. We can write

$$\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \mathbf{E}_1(\mathbf{r}, t) + \mathbf{E}_2(\mathbf{r}, t) . \qquad (5.47)$$

Importantly, no matter how close to being monochromatic the two sources are, they are always uncorrelated. To see this, consider the expression $\langle E_{1i}(\mathbf{r}_1, t)E_{2j}(\mathbf{r}_2, t+\tau)\rangle$. The key observation here is that the time t can be arbitrarily large compared to the characteristic time scale of oscillations of each field. Even if the fields start off as completely correlated at some instance in the past, small deviations from monochromaticity will eventually make the probably distribution of the averaged quantity to be symmetric about the origin. In other words, there is no physical reason why the product $E_{1i}(\mathbf{r}_1, t)E_{2j}(\mathbf{r}_2, t + \tau)$ would be positive rather than negative at large times t when the memory of any initial conditions has vanished. Consequently, the above average is zero. The only exception from this conclusion is the case of strict monochromaticity, which is not achievable in practice. Of course, the more monochromatic the two sources are, the longer one needs to wait until any initial correlation dies off. However, in the optical spectral range, this wait time is no larger than fraction of a second, even for the most monochromatic sources that are currently available. We will therefore proceed under the assumption that the coherence matrix of the incident field is

$$\Gamma_{\text{inc}}(\mathbf{r}_1, \mathbf{r}_2; \tau) = \Gamma_1(\mathbf{r}_1, \mathbf{r}_2; \tau) + \Gamma_2(\mathbf{r}_1, \mathbf{r}_2; \tau) , \qquad (5.48)$$

where Γ_1 and Γ_2 are the coherence matrices for $\mathbf{E}_1$ and $\mathbf{E}_2$. It immediately follows from (5.37) that absorption, scattering and extinction of independent incident fields are additive:

$$Q_* = Q_*^{(1)} + Q_*^{(2)} \ . \tag{5.49}$$

Here $Q_*^{(1)}$ is the absorbed, scattered or extinguished power that would have existed if only source 1 was on, etc.

We can expand this analysis to any number of independent sources. In this case, additivity can be expressed as

$$Q_* = \sum_{\mu} Q_*^{(\mu)} \ . \tag{5.50}$$

The quantities labeled by μ correspond to the case when only μ-th source is on. This formula is similar to (4.97), which was obtained under the assumption that different sources have non-overlapping spectra. Now we have shown that additivity of extinguished, absorbed and scattered powers holds in a more general setting. What is required for additivity is that the sources of different field components are physically independent, which includes the case of non-overlapping spectra as a special case.

Note that it is possible to use the same physical source to create two completely uncorrelated incident fields. For this to be possible, the source must have a finite and sufficiently wide spectrum. Then we can split the incident field using beam splitters and pass each resulting beam through spectral filters with non-overlapping windows. In this case, the analysis of Sec. 4.8 will apply. Therefore, the condition of physical independence of sources is sufficient but not required for additivity.

Notwithstanding what is written above, there exist no fundamental laws prohibiting interference of fields produced by independent sources. It is definitely possible to observe beats of two quasi-monochromatic narrow spectral lines generated by two different sources. The idea was published as early as in 1947 by Forrester, Parkins and Gerjuoy [63] and independently by Gorelik [17], and an experimental demonstration was published in 1955 [64]. With modern experimental techniques, it is relatively easy to observe interference fringes from two different lasers [65], but the pattern is not stable and must be observed or photographed over sufficiently short periods of time. These effects however have no influence on absorbed, scattered or extinguished powers as long as they are properly time-averaged.

5.6 Weak scattering

The weak scattering regime is obtained when the target has a low contrast relative to vacuum as quantified by the coupling function $\kappa(\mathbf{r}, \omega)$, and is also in some sense small. The requirements of small contrast and small size counter-balance each other. That is, the smaller is the contrast, the larger is the size for which the approximation is still accurate, although stating the exact relation between the two parameters is difficult. If the contrast is sufficiently small, the target size can be considerably larger

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[17] Doklady Akademii Nauk (DAN), **43**, 46 (1947). Not translated.

than what is required for the quasi-static approximation to set in. Therefore, the weak scattering regime is not inherently quasi-static. Formally, the weak scattering approximation can be obtained as an expansion in powers of the operator $\mathtt{V}(\omega)$, which is defined in terms of $\kappa(\mathbf{r}, \omega)$ in (4.79). Keeping only the first-order term in this expansion is known as the first Born approximation. We have already stated this approximation for monochromatic fields in Sec. 4.2.1 and more formally in Sec. 4.7 where we have also derived the second-order terms. In this subsection we state the first Born approximation for partially-coherent fields. We will derive the result for extinction only, as scattering is zero and absorption is equal to extinction to first order in $\mathtt{V}(\omega)$.

The first Born approximation for the T-matrix is obtained by keeping only the first term in the right-hand side of (5.20), which yields

$$\mathtt{T}(\mathbf{r}, \mathbf{r}'; \omega) \approx \kappa(\mathbf{r}, \omega) \delta(\mathbf{r} - \mathbf{r}') \mathtt{I}_3 , \tag{5.51}$$

where $\kappa(\mathbf{r}, \omega)$ is defined in (4.18). Using this approximation, we can simplify (5.31b) as

$$q_{\text{ext}}(\mathbf{r}, \mathbf{r}') \approx \frac{\delta(\mathbf{r} - \mathbf{r}')}{\pi} \int_0^\infty \left[\text{Im} \, \kappa(\mathbf{r}, \omega) \right] S_{\text{inc}}(\mathbf{r}, \omega) \, \omega \mathrm{d}\omega . \tag{5.52}$$

The density of extinction in the above expression is local. We therefore have for the total extinguished power

$$Q_{\text{ext}} \approx \frac{1}{\pi} \int_0^\infty \omega \mathrm{d}\omega \int_{\mathbb{V}} \mathrm{d}^3 r \, S_{\text{inc}}(\mathbf{r}, \omega) \, \text{Im}[\kappa(\mathbf{r}, \omega)] . \tag{5.53}$$

The expression is especially simple if the target is homogeneous in $\mathbb{V}$ so that $\kappa(\mathbf{r}, \omega) = \kappa_0(\omega)$. We then have

$$Q_{\text{ext}} \approx \frac{1}{\pi} \int_0^\infty U_{\text{inc}}(\omega) \, \text{Im}[\kappa_0(\omega)] \, \omega \mathrm{d}\omega , \tag{5.54}$$

where

$$U_{\text{inc}}(\omega) = \int_{\mathbb{V}} S_{\text{inc}}(\mathbf{r}, \omega) \mathrm{d}^3 r = \sum_n w_n(\omega) , \tag{5.55}$$

and $w_n(\omega)$ are the eigenvalues of the operator $\mathtt{W}(\omega)$ defined in (5.6b). We can say that $U_{\text{inc}}(\omega)$ is the integral spectral density of the incident field in $\mathbb{V}$.

As mentioned above, within the first Born approximation we have

$$Q_{\text{abs}} \approx Q_{\text{ext}} , \quad Q_{\text{sca}} \approx 0 . \tag{5.56}$$

This makes physical sense as the approximation is based on the assumption of negligible scattering. Non-vanishing scattered power can be obtained to second order in $V(\omega)$. However, computing this correction is only justified if it is in some sense small. Otherwise, accounting for the second-order term may not be a useful approximation. The reason is that, typically, there exists only a small region of parameters in which the first Born approximation is significantly inaccurate (and therefore can be corrected) but the expansion of the T-matrix in powers of $V(\omega)$ converges. Only if parameters of the problem are in this region, accounting for the second-order term is meaningful. But this is rarely known *a priori* without solving the problem non-perturbatively.

5.7 Quasi-monochromatic field

We can assume that the field is quasi-monochromatic if it is sufficient to compute the T-matrix $T(\omega)$ or the scattering operator $\Sigma(\omega)$ at just one frequency ω_0. In this sub-section, we will examine the conditions of applicability of the quasi-monochromatic approximation and derive expressions for the absorbed, scattered and extinguished powers assuming these conditions hold.

Consider the case when the spectrum of the incident field is narrow and concentrated around some frequency ω_0. Let the essential support of the cross-spectral density lie in the interval $[\omega_0 - \Delta/2, \omega_0 + \Delta/2]$ where $\omega_0 > 0$. This means that, for any $\omega > 0$ outside of this interval and $\mathbf{r}, \mathbf{r}' \in \mathbb{V}$, $W(\mathbf{r}_1, \mathbf{r}_2; \omega)$ is negligibly small [18]. A more quantitative definition of the essential support is given in Sec. 5.9.1 below. We require that $T(\omega)$ be almost independent of ω in the above interval of frequencies. Whether this condition holds depends on many factors and is harder to satisfy when the target has narrow spectral features known as electromagnetic resonances. However, one heuristic condition of applicability that we can state is

$$D \ll \ell_{\text{coh}} = \pi c / \Delta , \quad D = \max_{\mathbf{r}, \mathbf{r}' \in \mathbb{V}} |\mathbf{r} - \mathbf{r}'| . \tag{5.57}$$

Here D is the spatial extent of the target and ℓ_{coh} is the coherence length (sometimes defined with the factor of 2π rather than π).

The condition (5.57) is neither sufficient nor necessary for the quasi-monochromatic approximation to hold, but is rather a heuristic rule of thumb. In a practical computation, one should verify that $T(\omega)$ is almost constant and has no obvious spectral features in the interval $[\omega_0 - \Delta/2, \omega_0 + \Delta/2]$. We proceed under the assumption that

$$\omega \, T(\mathbf{r}_1, \mathbf{r}_2; \omega) \approx \omega_0 \, T(\mathbf{r}_1, \mathbf{r}_2; \omega_0) \quad \text{if} \quad \omega_0 - \Delta/2 \leq \omega \leq \omega_0 + \Delta/2 \tag{5.58}$$

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[18] The cross-spectral density, as defined in this chapter, satisfies $W(-\omega) = W^*(\omega)$. Therefore, there is also a spectral peak at $\omega = -\omega_0$. However, we express all measurable quantities as integrals over positive frequencies.

is a sufficiently accurate approximation. To compute Q_{ext}, it is convenient to use the formulas of Sec. 5.4.2 where we have explicitly isolated the imaginary part of the T-matrix. Specifically, applying the approximation (5.58) to (5.43), we obtain

$$Q_{\text{ext}} \approx \frac{\omega_0}{\pi} \, \text{Tr} \int\limits_{\mathbb{V}} d^3r \int\limits_{\mathbb{V}} d^3r' \left[\mathcal{W}_{\text{inc}}^{(\text{r})}(\mathbf{r}, \mathbf{r}') \, \mathsf{T}^{(\text{i})}(\mathbf{r}', \mathbf{r}; \omega_0) \right] \, . \qquad (5.59)$$

where

$$\mathcal{W}_{\text{inc}}^{(\text{r})}(\mathbf{r}, \mathbf{r}') = \int\limits_{0}^{\infty} \mathsf{W}_{\text{inc}}^{(\text{r})}(\mathbf{r}, \mathbf{r}'; \omega) \, d\omega \, . \qquad (5.60)$$

Since the cross-spectral density and the coherence matrix are Fourier transforms of each other, we know that

$$\int\limits_{-\infty}^{\infty} \mathsf{W}_{\text{inc}}(\mathbf{r}, \mathbf{r}'; \omega) \, d\omega = 2\pi \, \Gamma_{\text{inc}}(\mathbf{r}, \mathbf{r}'; 0) \, , \qquad (5.61)$$

which implies that

$$\mathcal{W}_{\text{inc}}^{(\text{r})}(\mathbf{r}, \mathbf{r}') = \pi \, \Gamma_{\text{inc}}(\mathbf{r}, \mathbf{r}'; 0) \, . \qquad (5.62)$$

We therefore have

$$Q_{\text{ext}} \approx \omega_0 \, \text{Tr} \int\limits_{\mathbb{V}} d^3r \int\limits_{\mathbb{V}} d^3r' \, \Gamma_{\text{inc}}(\mathbf{r}, \mathbf{r}'; 0) \, \mathsf{T}^{(\text{i})}(\mathbf{r}', \mathbf{r}; \omega_0) \, . \qquad (5.63)$$

Here Γ_{inc} is purely real and the operation of taking the imaginary part has been applied only to T.

Note that similar results cannot be obtained for the absorbed and scattered powers without additional assumptions or approximations. The reason is that neither of the operators $\Pi_{\text{sca}}(\omega)$ and $\Pi_{\text{abs}}(\omega)$ or $\mathsf{H}_{\text{sca}}(\omega)$ and $\mathsf{H}_{\text{sca}}(\omega)$ are symmetric. The latter two operators are complex Hermitian, but this is insufficient for our purposes. Consequently, we cannot compute the positive frequency integral of the type (5.43) as easily. Extinction is in this respect unique as the operator $\mathsf{H}_{\text{ext}}(\omega)$ is real symmetric, which is what has really facilitated the above mathematical development. The forthcoming discussion is, therefore, focused on extinction.

Expression (5.63) is, in general, different from the corresponding result for the strictly monochromatic field, i.e., (4.74c) or (4.75c). In particular, it allows for different Cartesian component of the field to be partially coherent or even completely uncorrelated. For example, a quasi-monochromatic plane wave can be unpolarized whereas a strictly monochromatic plane wave is always polarized. Unlike its strictly monochromatic counter-parts, formula (5.63) allows for consideration of unpolarized light. However, (5.63) can describe strictly monochromatic waves as a limiting case.

Indeed, assume that the incident field is of the form $\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \text{Re}[\mathbf{E}_{\text{inc}}(\mathbf{r})e^{-i\omega_0 t}]$ where $\mathbf{E}_{\text{inc}}(\mathbf{r})$ is a time-independent complex amplitude. We have in this case (i.e., see (7.22) below)

$$\Gamma_{\text{inc}}(\mathbf{r}, \mathbf{r}'; 0) = \frac{1}{2}\text{Re}\left[\mathbf{E}_{\text{inc}}^*(\mathbf{r}) \otimes \mathbf{E}_{\text{inc}}(\mathbf{r}')\right] . \tag{5.64}$$

Substituting this expression into (5.63) we obtain after a few straightforward algebraic manipulations

$$Q_{\text{ext}} \approx \frac{\omega_0}{2} \text{Re} \int_{\mathbb{V}} d^3r \int_{\mathbb{V}} d^3r' \, \mathbf{E}_{\text{inc}}^*(\mathbf{r}) \cdot \mathsf{T}^{(i)}(\mathbf{r}, \mathbf{r}'; \omega_0)\mathbf{E}_{\text{inc}}(\mathbf{r}') . \tag{5.65}$$

This result is equivalent to (4.74c). To see that this is indeed so, we must take the symmetry $\mathsf{T}(\mathbf{r}, \mathbf{r}'; \omega_0)$ into account, as it was done in Sec. 5.4.2 above. In the case of an incident plane wave described mathematically by $\mathbf{E}_{\text{inc}}(\mathbf{r}) = \mathbf{A}e^{i\mathbf{k}\cdot\mathbf{r}}$, (5.65) is reduced to the previously-derived expression (4.72).

5.8 Plane wave

A plane wave can be characterized completely by two scalar functions of a single scalar variable. As a result, the general expressions for the absorbed, scattered and extinguished powers are greatly simplified. In the case of incident plane waves, it is common to consider the scattering problem in frequency domain assuming monochromatic illumination; then the result is extended to non-monochromatic fields as a spectral integral. For example, the total power of the solar radiation [19] extinguished by a water droplet in the atmosphere can be computed as an integral of the solar spectrum multiplied by the extinction spectrum of the droplet, where the latter is computed from the Mie theory for monochromatic fields. In this section, we will give a rigorous mathematical justification for this approach paying particular attention to the effects of polarization and to precise definitions of the quantities that should be used in the spectral integrals. In particular, we will demonstrate rigorously the principle of spectral additivity of extinction, which is used implicitly in the above example.

5.8.1 Polychromatic plane wave

An incident plane wave propagating along the Z-axis can be written in the most general form as

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[19] Strictly speaking, solar radiation is not a plane wave since the solid angle subtended by the Sun when viewed from Earth is not vanishingly small.

$$\mathbf{E}_{\text{inc}}(z, t) = \mathbf{e}_x \mathcal{E}_x(t - z/c) + \mathbf{e}_y \mathcal{E}_y(t - z/c) , \tag{5.66}$$

where $\mathbf{e}_x$ and $\mathbf{e}_y$ and $\mathbf{e}_z$ are unit vectors along the X and Y-axes and $\mathcal{E}_x(t)$, $\mathcal{E}_y(t)$ are two arbitrary real, scalar, stationary stochastic processes. The statistical properties of the plane wave are completely characterized by the following three correlation functions:

$$g_x(\tau) = \langle \mathcal{E}_x(t) \mathcal{E}_x(t + \tau) \rangle , \tag{5.67a}$$

$$g_y(\tau) = \langle \mathcal{E}_y(t) \mathcal{E}_y(t + \tau) \rangle , \tag{5.67b}$$

$$h(\tau) = \langle \mathcal{E}_x(t) \mathcal{E}_y(t + \tau) \rangle . \tag{5.67c}$$

It follows from the above definitions that $g_{x,y}(-\tau) = g_{x,y}(\tau)$. However, there is no such requirement for $h(\tau)$. Note that all functions in (5.67) are real.

The incident wave (5.66) can be linearly polarized, have a mixed polarization or be completely unpolarized. If the electric field is confined entirely to one plane, the wave is linearly polarized. We can assume without loss of generality that $\mathcal{E}_y = 0$ in this case and the plane of polarization is XZ. We can also envisage a situation in which the field is not confined to one plane and $g_x(\tau) \neq g_y(\tau)$. In this case, polarization is mixed. The X- and Y-polarized components of the wave can have different spectra and the off-diagonal correlation function $h(\tau)$ can be rather arbitrary subject to the constraints that follow from the Schwartz inequality such as $h^2(\tau) \leq g_x(0)g_y(0)$. Finally, if $g_x(\tau) = g_y(\tau)$ and $h(\tau) = 0$, we have the case of completely unpolarized wave. Note that the statistical properties of an unpolarized wave are invariant under rotation of the reference frame about the Z-axis. Circular (more generally, elliptic) polarization is only possible for a quasi-monochromatic wave with a well-defined central frequency ω_0. This case is considered in Sec. 5.8.2 below.

In the reference frame XYZ introduced above, the coherence matrix of the incident plane wave (5.66) is of the form

$$\Gamma_{\text{inc}}(z, z'; \tau) = \begin{bmatrix} g_x(\xi) & h(\xi) \\ h(-\xi) & g_y(\xi) \end{bmatrix} , \quad \xi = \tau + \frac{z - z'}{c} . \tag{5.68}$$

Since the field (5.66) is transverse (has zero projection onto the Z-axis), we can work with Γ_{inc} as with a 2×2 matrix. The corresponding cross-spectral density can be expressed in terms of temporal Fourier transforms of $g_x(t)$, $g_y(t)$ and $h(t)$ as

$$\mathbb{W}(z, z'; \omega) = e^{\mathrm{i}k(z'-z)} \, \mathbb{J}(\omega) , \quad k = \omega/c , \tag{5.69}$$

where

$$\mathbb{J}(\omega) = \begin{bmatrix} g_x(\omega) & h(\omega) \\ h^*(\omega) & g_y(\omega) \end{bmatrix} \tag{5.70}$$

is the polarization matrix, sometimes referred to (in a somewhat different context) as the Jones matrix . The factorization of the spatial dependence of the cross spectral density, as expressed by the product of exponents $e^{\mathrm{i}kz'}e^{-\mathrm{i}kz}$, is a significant

simplification, which is specific to plane waves. It is important to keep in mind that the statistical properties of the incident field are characterized by the function $\mathtt{J}(\omega)$; knowledge of a single sample of this function such as $\mathtt{J}(\omega_0)$ is insufficient.

We can now compute the extinguished power. To this end, we substitute the cross-spectral density (5.69) into (5.31), integrate the result over $\mathbf{r}$ and $\mathbf{r}'$ according to (5.23) and recall the definition of spatial Fourier transform of the T-matrix (4.52). These steps result in

$$Q_{\text{ext}} = \frac{1}{\pi} \, \text{Im Tr} \int_0^{\infty} \left[\mathtt{T}(\mathbf{k}, \mathbf{k}; \omega) \, \mathtt{J}^{\mathrm{T}}(\omega) \right] \omega \mathrm{d}\omega \tag{5.71a}$$

$$= \frac{1}{\pi} \, \text{Im Tr} \int_0^{\infty} \left[\mathtt{J}(\omega) \mathtt{T}^{\mathrm{T}}(\mathbf{k}, \mathbf{k}; \omega) \right] \omega \mathrm{d}\omega \,, \tag{5.71b}$$

where

$$\mathbf{k} = \mathbf{e}_z \, \omega/c \,. \tag{5.72}$$

Here $\mathbf{k}$ is the wave vector of a hypothetical monochromatic plane wave of frequency ω propagating in the Z-direction. Thus, $\mathtt{T}(\mathbf{k}, \mathbf{k}; \omega)$ is the temporal and spatial Fourier transform of the time-domain, real-space T-matrix:

$$\mathtt{T}(\mathbf{k}, \mathbf{k}; \omega) = \int_0^{\infty} e^{\mathrm{i}\,\omega\tau} \mathrm{d}\tau \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \, e^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \, \mathtt{T}(\mathbf{r}, \mathbf{r}'; \tau) \, e^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}'} \,. \tag{5.73}$$

Note that $\mathbf{k}$ depends on ω as indicated in (5.72). The expression (5.71a) was obtained from (5.31a) and (5.71b) was obtained from (5.31b). The two formulas are equivalent since $\text{Tr}[\mathtt{A}\mathtt{B}^{\mathrm{T}}] = \text{Tr}[\mathtt{A}^{\mathrm{T}}\mathtt{B}]$. However, note that, in order to derive (5.71b), we used the symmetry property

$$\mathtt{T}(\mathbf{k}, \mathbf{k}; \omega) = \mathtt{T}^{\mathrm{T}}(-\mathbf{k}, -\mathbf{k}; \omega) \,, \tag{5.74a}$$

which follows directly from the definition (5.73). In particular, the diagonal elements of $\mathtt{T}(\mathbf{k}, \mathbf{k}; \omega)$ are invariant with respect to reverting the propagation direction of the incident wave:

$$T_{xx}(\mathbf{k}, \mathbf{k}; \omega) = T_{xx}(-\mathbf{k}, -\mathbf{k}; \omega) \,, \quad T_{yy}(\mathbf{k}, \mathbf{k}; \omega) = T_{yy}(-\mathbf{k}, -\mathbf{k}; \omega) \,, \tag{5.74b}$$

whereas the off-diagonal elements obey

$$T_{xy}(\mathbf{k}, \mathbf{k}; \omega) = T_{yx}(-\mathbf{k}, -\mathbf{k}; \omega) \,. \tag{5.74c}$$

Equations (5.71) provide the most general result for extinction of a plane, partially-coherent wave. Similar expressions can be obtained for absorption and scattering

by replacing $\Pi_{\text{ext}}(\omega) = \mathtt{T}(\omega)$ with $\Pi_{\text{abs}}(\omega)$ or $\Pi_{\text{sca}}(\omega)$, where the Π-operators are defined in (4.86).

It is also instructive to derive an expression for Q_{ext} using the eigen-decomposition of $\mathtt{W}(\omega)$ defined in equations (5.6) of Sec. 5.2. The eigenfunctions and eigenvalues of $\mathtt{W}_{\text{inc}}(\omega)$ can be easily found due to separability of the kernel (5.69). We have

$$\boldsymbol{F}_n(z, \omega) = \frac{1}{\sqrt{V[\mathbb{V}]}} \, e^{-ikz} \, \boldsymbol{f}_n(\omega) \, , \quad w_n(\omega) = V[\mathbb{V}] \, \lambda_n(\omega) \, , \qquad (5.75a)$$

where $\boldsymbol{f}_n(\omega)$ and $\lambda_n(\omega)$ are the eigenvectors and eigenvalues of the matrix $\mathtt{J}(\omega)$, viz,

$$\mathtt{J}(\omega)\boldsymbol{f}_n(\omega) = \lambda_n(\omega)\boldsymbol{f}_n(\omega) \, , \quad \boldsymbol{f}_m^*(\omega) \cdot \boldsymbol{f}_n(\omega) = \delta_{mn} \, . \qquad (5.75b)$$

As $\mathtt{J}(\omega)$ is a 2×2 Hermitian matrix, it has two real eigenvalues $\lambda_\pm(\omega)$ and two orthogonal eigenvectors $\boldsymbol{f}_\pm(\omega)$ at each frequency:

$$\lambda_\pm(\omega) = \frac{g_x(\omega) + g_y(\omega)}{2} \pm \sqrt{\left(\frac{g_x(\omega) - g_y(\omega)}{2}\right)^2 + |h(\omega)|^2} \, , \qquad (5.76a)$$

$$\boldsymbol{f}_\pm(\omega) = \frac{1}{Z_\pm(\omega)} \begin{bmatrix} h(\omega) \\ \lambda_\pm(\omega) - g_x(\omega) \end{bmatrix} \, , \qquad (5.76b)$$

where

$$Z_\pm^2(\omega) = |h(\omega)|^2 + [\lambda_\pm(\omega) - g_x(\omega)]^2 \, . \qquad (5.76c)$$

We can use this eigen-decomposition to write $\mathtt{J}(\omega)$ as a sum of two separable and therefore zero-determinant matrices

$$\mathtt{J}(\omega) = \mathtt{J}_+(\omega) + \mathtt{J}_-(\omega) \, , \qquad (5.77a)$$

where

$$\mathtt{J}_\pm(\omega) = \lambda_\pm(\omega) \, \boldsymbol{f}_\pm(\omega) \otimes \boldsymbol{f}_\pm^*(\omega) \, . \qquad (5.77b)$$

A straightforward algebraic manipulation yields the following result for $\mathtt{J}_\pm(\omega)$:

$$\mathtt{J}_\pm(\omega) = \frac{\lambda_\pm(\omega)}{2\lambda_\pm(\omega) - g_x(\omega) - g_y(\omega)} \begin{bmatrix} \lambda_\pm(\omega) - g_y(\omega) & h(\omega) \\ h^*(\omega) & \lambda_\pm(\omega) - g_x(\omega) \end{bmatrix} \, . \quad (5.78)$$

Note that (5.78) has a well-defined limit when $2\lambda_\pm(\omega) - g_x(\omega) - g_y(\omega) \to 0$. The above singularity occurs if $g_x(\omega) = g_y(\omega) = g(\omega)$ and $h(\omega) = 0$. In this case,

$$\mathtt{J}_+(\omega) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \, , \quad \mathtt{J}_-(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \, . \qquad (5.79)$$

It can be seen that, in all cases, $\det[\mathsf{J}_\pm(\omega)] = 0$. This follows from the definition (5.77b) and can be verified directly by observing that the eigenvalues $\lambda_\pm(\omega)$ satisfy the characteristic equation

$$[g_x(\omega) - \lambda(\omega)] \, [g_y(\omega) - \lambda(\omega)] - |h(\omega)|^2 = 0 \; . \tag{5.80}$$

Substituting $\boldsymbol{F}_n(\mathbf{r}, \omega)$ from (5.75a) into (5.34) and the result into (5.33) yields

$$Q_{\text{ext}} = \int\limits_0^\infty [\mathcal{Q}_{\text{ext}}^{(+)}(\omega) + \mathcal{Q}_{\text{ext}}^{(-)}(\omega)]\mathrm{d}\omega \; , \tag{5.81a}$$

where

$$\mathcal{Q}_{\text{ext}}^{(\pm)}(\omega) = \frac{\omega}{\pi} \operatorname{Im} \operatorname{Tr} \, \left[\mathsf{T}(\mathbf{k}, \mathbf{k}; \omega) \mathsf{J}_\pm^{\mathsf{T}}(\omega) \right] \tag{5.81b}$$

$$= \frac{\omega}{\pi} \operatorname{Im} \operatorname{Tr} \, \left[\mathsf{J}_\pm(\omega) \, \mathsf{T}^{\mathsf{T}}(\mathbf{k}, \mathbf{k}; \omega) \right] \; . \tag{5.81c}$$

As above, we have used the curly symbol $\mathcal{Q}$ to indicate that the respective quantity is not power (in this instance, $\mathcal{Q}$ has the units of energy). The result (5.81) is very similar to (5.71), the only difference being that we have decomposed $\mathsf{J}(\omega)$ as $\mathsf{J}_+(\omega) + \mathsf{J}_-(\omega)$. There exist, of course, an infinite number of ways in which a 2×2 matrix can be decomposed into a sum of two matrices. The decomposition (5.77) is however special (and unique) because $\mathsf{J}_\pm(\omega)$ have zero determinants. As a result, each of these matrices is equal to the Jones matrix of some hypothetical monochromatic plane wave. We emphasize that this equivalence cannot hold for $\mathsf{J}(\omega)$ itself as its determinant is, in general, non-zero, whereas the Jones matrices of monochromatic plane waves have zero determinants. This observation is conceptually important and it sheds additional light onto the physical significance of the eigen-decomposition of $\mathsf{W}(\omega)$.

To elaborate further on this point, consider a monochromatic incident plane wave of the form

$$\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \operatorname{Re}[\boldsymbol{A} \, e^{\mathrm{i}\,(\mathbf{k}\cdot\mathbf{r} - \omega t)}] \; . \tag{5.82}$$

The polychromatic incident wave (5.66) is not a superposition of such monochromatic waves. However, we will show momentarily that the extinguished power due to (5.66) can be written as a spectral integral involving extinguished powers due to monochromatic incident fields of the form (5.82). In fact, (5.81b) is a spectral integral of this kind. Indeed, the extinguished power for an incident wave of the form (5.82) is given by

$$Q_{\text{ext}} = \frac{\omega}{2} \operatorname{Im}[\boldsymbol{A}^* \cdot \mathsf{T}(\mathbf{k}, \mathbf{k}; \omega)\boldsymbol{A}] \; . \tag{5.83}$$

The is a special case of the more general result (4.55), which was derived in Sec. 4.5 above. We can re-write (5.83) identically as

$$Q_{\text{ext}}(\omega) = \frac{\omega}{2} \, \text{Im Tr} \, \left[\mathsf{T}(\mathbf{k}, \mathbf{k}; \omega) \, \mathsf{J}_{\text{mon}}^{\mathsf{T}} \right] \tag{5.84a}$$

$$= \frac{\omega}{2} \, \text{Im Tr} \, \left[\mathsf{J}_{\text{mon}} \, \mathsf{T}^{\mathsf{T}}(\mathbf{k}, \mathbf{k}; \omega) \right] \, . \tag{5.84b}$$

where

$$\mathsf{J}_{\text{mon}} = \begin{bmatrix} A_x^* A_x & A_x^* A_y \\ A_y^* A_x & A_y^* A_y \end{bmatrix} \, . \tag{5.85}$$

is the Jones matrix for the monochromatic plane wave (5.82). As mentioned above, $\det[\mathsf{J}_{\text{mon}}(\omega)] = 0$, which is a general property of all monochromatic plane waves. Consequently, we can always find such an amplitude A that $\mathsf{J}_{\text{mon}} = \mathsf{J}_{\pm}(\omega)$ (for both choices of sign). The solution to this problem is

$$\boldsymbol{A}_{\pm} = \sqrt{\lambda_{\pm}(\omega)} \boldsymbol{f}_{\pm}^*(\omega) \, . \tag{5.86}$$

Thus, we can interpret $\mathcal{Q}_{\text{ext}}^{(\pm)}(\omega)$ in (5.81) as the power extinguished by the target due to an incident plane wave of the form

$$\mathbf{E}_{\pm}(\mathbf{r}, t; \omega) = \text{Re}[\sqrt{\lambda_{\pm}(\omega)} \boldsymbol{f}_{\pm}^*(\omega) \, e^{i\,\omega(z/c-t)}] \, , \tag{5.87}$$

where we have used $\mathbf{k} \cdot \mathbf{r} = \omega z/c$. Here the extra label ω in the notation $\mathbf{E}_{\pm}(\mathbf{r}, t; \omega)$ indicates that we consider a family of two monochromatic waves (two polarization modes at each frequency corresponding to $+$ or $-$ sign in the index) whose amplitudes depend on ω. Note that $\mathbf{E}_{\pm}(\mathbf{r}, t; \omega)$ is a *hypothetical* incident wave. In particular, it does not have the units of electric field. However, the total extinguished power for the polychromatic incident wave (5.66) is given by a spectral integral of extinguished powers for such hypothetical waves, summed over the polarization states at each frequency.

5.8.2 Quasi-monochromatic plane wave

In Sec. 5.7, we considered general quasi-monochromatic incident field for which the essential support of the power spectrum lies in the interval $[\omega_0 - \Delta/2, \omega_0 + \Delta/2]$. The main formula derived in that section is (5.63). However, using this without a specific model for the coherence matrix $\Gamma_{\text{inc}}(\mathbf{r}, \mathbf{r}; 0)$ is problematic. We need to know *something* about the spatial coherence of the incident field to apply (5.63). One approach is to use various physical models for the coherence matrix. For example, the Lorentz model of power spectrum, which is, essentially, a perturbation over the monochromatic limit, is of the form

$$g_{x,y}(\tau) = I_{x,y} \cos(\omega_0 \tau) e^{-\gamma|\tau|} \, , \tag{5.88}$$

where $\gamma \ll \omega_0$. In addition, we need the function $h(\tau)$. The Lorentz model has little to say about the latter. To describe elliptic polarization, we can assume, for

example, that $\mathcal{E}_y(t) = \beta \mathcal{E}_x(t - s)$, where s is a fixed time shift. This would result in $h(\tau) = \beta g_x(\tau + s)$ and $g_y(\tau) = \beta^2 g_x(\tau)$. With these assumptions, it is possible to develop the expression (5.63) and obtain a number of useful approximations. However, the above model for $h(\tau)$ is not general and the results will depend explicitly on the mathematical form of the spectrum (not only on the assumption that it is narrow).

Here we will follow a more general and a more phenomenological approach by postulating that the Jones matrix in (5.70) can be approximated for quasi-monochromatic field as

$$\mathsf{J}(\omega) \approx 2\pi \, \mathsf{J}_0 \, \delta(\omega - \omega_0) \, . \tag{5.89}$$

Substituting this approximation into (5.71), we obtain

$$Q_{\mathrm{ext}} = 2\omega_0 \, \mathrm{Im \, Tr} \, \left[\mathsf{T}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) \, \mathsf{J}_0^{\mathsf{T}} \right] = 2\omega_0 \, \mathrm{Im \, Tr} \, \left[\mathsf{J}_0 \, \mathsf{T}^{\mathsf{T}}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) \right] \, , \tag{5.90}$$

where $\mathbf{k}_0 = (\omega_0/c)\mathbf{e}_z$. The matrix J_0 is often expressed in terms of the Stokes parameters of the incident wave (I, Q, U, V) as follows:

$$\mathsf{J}_0 = \left[\begin{array}{cc} I + Q & U + \mathrm{i}\,V \\ U - \mathrm{i}\,V & I - Q \end{array} \right] \, . \tag{5.91}$$

The Stokes parameters can be introduced in a less formal manner as follows. Let the functions $\mathcal{E}_{x,y}(t)$ describing a quasi-monochromatic plane wave be of the form

$$\mathcal{E}_{x,y}(t) = \mathrm{Re}[A_{x,y}(t)e^{-\mathrm{i}\,\omega_0 t}] \, , \tag{5.92}$$

where $A_{x,y}(t)$ are "slow" functions of time, by which we mean that they change on time scales much larger than $2\pi/\omega_0$. In practice, the separation of time scales can be of many orders of magnitude. We then have for the coherence matrix

$$\Gamma_{\mathrm{inc}}(z, z'; \tau) \approx \left[\begin{array}{cc} \langle A_x^*(t)A_x(t + \xi) \rangle & \langle A_x^*(t)A_y(t + \xi) \rangle \\ \langle A_y^*(t)A_x(t + \xi) \rangle & \langle A_y^*(t)A_y(t + \xi) \rangle \end{array} \right] e^{-\mathrm{i}\,\omega_0 \xi} \, , \tag{5.93}$$

where, as before, $\xi = \tau + (z - z')/c$ and we have assumed that the time-averages of fast-oscillating terms are zero. Now we compute the power spectrum as

$$\mathsf{W}_{\mathrm{inc}}(z, z'; \omega) = \int_{-\infty}^{\infty} \left[\begin{array}{cc} \langle A_x^*(t)A_x(t + \xi) \rangle & \langle A_x^*(t)A_y(t + \xi) \rangle \\ \langle A_y^*(t)A_x(t + \xi) \rangle & \langle A_y^*(t)A_y(t + \xi) \rangle \end{array} \right] e^{\mathrm{i}\,(\omega\tau - \omega_0\xi)} \mathrm{d}\tau$$

$$= e^{\mathrm{i}\,(z' - z)/c} \int_{-\infty}^{\infty} \left[\begin{array}{cc} \langle A_x^*(t)A_x(t + \xi) \rangle & \langle A_x^*(t)A_y(t + \xi) \rangle \\ \langle A_y^*(t)A_x(t + \xi) \rangle & \langle A_y^*(t)A_y(t + \xi) \rangle \end{array} \right] e^{\mathrm{i}\,(\omega - \omega_0)\tau} \mathrm{d}\tau \, . \tag{5.94}$$

Since $A_{x,y}(t)$ are slow functions of t, their correlators also change on time scales much larger than $2\pi/\omega_0$. Correspondingly, we can set $\xi = 0$ inside the matrix. The approximation used here is to assume that the integral (5.94) converges (to a delta

function) while we still have $\langle A_i^*(t) A_j(t + \xi) \rangle \approx \langle A_i^*(t) A_j(t) \rangle$. We thus obtain

$$
\mathsf{W}_{\text{inc}}(z, z'; \omega) = 2\pi\, e^{\mathrm{i}\,(z'-z)/c} \left[\begin{array}{cc} \langle |A_x(t)|^2 \rangle & \langle A_x^*(t) A_y(t) \rangle \\ \langle A_y^*(t) A_x(t) \rangle & \langle |A_y(t)|^2 \rangle \end{array} \right] \delta(\omega - \omega_0)\,. \quad (5.95)
$$

Comparing to (5.69) and (5.89), we see that (5.95) agrees with these two equations if we set

$$
\mathsf{J}_0 = \left[\begin{array}{cc} \langle |A_x(t)|^2 \rangle & \langle A_x^*(t) A_y(t) \rangle \\ \langle A_y^*(t) A_x(t) \rangle & \langle |A_y(t)|^2 \rangle \end{array} \right]\,. \quad (5.96)
$$

This observation justifies the use of approximation (5.89). We then define the Stokes parameters as

$$
I = \frac{1}{2} \left(\langle |A_x(t)|^2 \rangle + \langle |A_y(t)|^2 \rangle \right)\,, \quad (5.97\text{a})
$$

$$
Q = \frac{1}{2} \left(\langle |A_x(t)|^2 \rangle - \langle |A_y(t)|^2 \rangle \right)\,, \quad (5.97\text{b})
$$

$$
U = \mathrm{Re}\, \langle A_x^*(t) A_y(t) \rangle\,, \quad (5.97\text{c})
$$

$$
V = \mathrm{Im}\, \langle A_x^*(t) A_y(t) \rangle\,. \quad (5.97\text{d})
$$

With these definitions, we recover the expression (5.91) for J_0.

The Stokes parameters are commonly used in optics to characterize the polarization states of plane waves. In particular, non-zero values of the parameter V correspond to elliptic polarization. The Stokes vectors of the form $I(1, 0, 0, 1)$ and $I(1, 0, 0, -1)$ correspond to right and left circular polarizations, respectively. Plane waves with $V = 0$ do not possess any "handiness". We refer the reader to the textbook [2] for further details, but would like to offer a word of caution. Namely, the Stokes parameters can be defined only for quasi-monochromatic waves. Indeed, there is no unique or reasonable way to define the slow amplitudes $A_{x,y}(t)$ for a broad spectrum. In practice, the condition of quasi-monochromaticity does not need to be strong and characterization of light beams in terms of Stokes vectors is quite robust to violations of the underlying assumption of quasi-monochromaticity. However, it is possible and in fact easy to find physical situations in which the use of Stokes parameters is not justified. This is the case, for example, for many kinds of unfiltered thermal radiation.

We now return to the expression for the extinguished power (5.90). Writing it out in components and using (5.91) for J_0, we obtain

$$
\begin{aligned}
Q_{\text{ext}} = 2\omega_0 \Big\{ &I\, \mathrm{Im}[\mathsf{T}_{xx}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) + \mathsf{T}_{yy}(\mathbf{k}_0, \mathbf{k}_0; \omega_0)] \\
&+Q\, \mathrm{Im}[\mathsf{T}_{xx}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) - \mathsf{T}_{yy}(\mathbf{k}_0, \mathbf{k}_0; \omega_0)] \\
&+U\, \mathrm{Im}[\mathsf{T}_{xy}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) + \mathsf{T}_{yx}(\mathbf{k}_0, \mathbf{k}_0; \omega_0)] \\
&+V\, \mathrm{Re}[\mathsf{T}_{yx}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) - \mathsf{T}_{xy}(\mathbf{k}_0, \mathbf{k}_0; \omega_0)] \Big\}\,. \quad (5.98)
\end{aligned}
$$

Targets for which $T_{yx}(\mathbf{k}_0, \mathbf{k}_0; \omega_0) - T_{xy}(\mathbf{k}_0, \mathbf{k}_0; \omega_0)$ can be different from zero are known as chiral. In the case of chiral targets, two incident plane waves with Stokes vectors $I(1, 0, 0, 1)$ and $I(1, 0, 0, -1)$ produce different extinguished powers, which is known as the effect of circular dichroism. Conversely, the extinguished power of chiral targets can change if we keep the Stokes parameters the same and change the direction of propagation, which is described mathematically by the substitution $\mathbf{k}_0 \rightarrow -\mathbf{k}_0$.

If the target can be superimposed onto its mirror image by a combination of translations and rotations, the target is non-chiral and the last term in (5.98) is identically zero. Moreover, for such targets it is possible to rotate the X and Y axes about the Z axis so that the third term is also zero. In such case, the off-diagonal elements of $T(\omega_0)$ are unimportant for extinction. Examples of non-chiral targets include spheres, truncated cylinders, cones, cuboids and many other regular shapes.

5.9 Quasi-static approximation

5.9.1 Assumptions and conditions of applicability

Generally, the quasi-static approximation is applicable to sub-wavelength targets. More specifically, we require that

$$\max_{\mathbf{r}, \mathbf{r}' \in \mathbb{V}; \; S_{\text{inc}}(\mathbf{r}, \omega) > S_{\min}} \left(\frac{\omega}{c} |\mathbf{r} - \mathbf{r}'| \right) \leq \varepsilon_1 \ll 1 , \qquad (5.99)$$

where ε_1 is a small dimensionless constant and $S_{\min} > 0$ is a threshold value for $S_{\text{inc}}(\mathbf{r}, \omega)$. Since most spectra never turn exactly to zero at any frequency, we would not be able to satisfy the condition (5.99) unless we define such a finite threshold. While there is no first-principle method to specify $S_{\min}$, we can determine it heuristically as the solution to the equation

$$\int_0^\infty d\omega \int_{\mathbb{V}} d^3r \; S_{\text{inc}}(\mathbf{r}, \omega) \, \Theta \left(S_{\min} - S_{\text{inc}}(\mathbf{r}, \omega) \right) = \varepsilon_2 \int_0^\infty d\omega \int_{\mathbb{V}} d^3r \; S_{\text{inc}}(\mathbf{r}, \omega) ,$$

$$(5.100)$$

where $\Theta(x)$ is the unit step function and $\varepsilon_2 \ll 1$ is another small dimensionless constant. For spectra that are concentrated around a given frequency ω_0, we can replace the condition $S_{\text{inc}}(\mathbf{r}, \omega) > S_{\min}$ with the much simpler condition $\omega_0 - \Delta/2 \leq \omega \leq \omega_0 + \Delta/2$, where Δ is the spectral width. For commonly-encountered Lorenzian spectra, the selection $\Delta = 12\gamma$, where γ is the width of the Lorenzian, approximately corresponds to $\varepsilon_2 = 0.01$ in (5.100) and, roughly, to a 1% relative error of the approximation.

Selecting the value of ε_1 in (5.99) is a more complicated matter. The quasi-static approximation requires that we neglect the time shifts due to retardation,

$|\mathbf{r} - \mathbf{r}'|/c$, in all equations of Sec. 5.3. It is correct to do so in the mathematical limit $\varepsilon_1 \to 0$. However, we should keep in mind that these equations also contain the coupling function $\kappa(\mathbf{r}, \tau)$. As a result, the value of ε_1 that is required to reach this limit depends on the properties of the target. For transparent dielectrics at optical frequencies, it is often sufficient to take $\varepsilon_1 \sim 0.1$. For high-conductivity metals below the plasma frequency, the requirement is much more stringent. A rough estimate can be obtained by considering a homogeneous spherical target of radius a for which an analytical solution to the electromagnetic problem is known. For the quasi-static approximation to be accurate, the amplitude of the induced electric dipole of the sphere must be much larger than the amplitudes of all other induced multipole moments, most importantly, of the magnetic dipole moment. If this condition holds, it is sufficient to take $\varepsilon_1 = \omega a/c$. Note that the magnetic moments should be small at all frequencies within the essential spectrum.

Above, we have stated the condition of target smallness. This is the only essential condition for the quasi-static approximation to be valid. However, to simplify the analysis, we will make two additional assumptions, which are not related to the target size. Namely, we will assume that

 (i) the target is spatially-uniform;
(ii) the sources of radiation are located in the far zone of the target, so that the incident field is also spatially-uniform over $\mathbb{V}$ at any instance of time.

A couple of obvious examples in which the quasi-static approximation can still be valid but the above two assumptions do not hold include multi-component targets or illumination in the Kretschmann geometry. While we do not cover these cases here, it is hoped that the reader who follows the derivations below will know how to make the required generalizations.

The assumption (i) means that the coupling function $\kappa(\mathbf{r}, \tau)$ or its Fourier transform $\kappa(\mathbf{r}, \omega)$ are independent of $\mathbf{r}$ in $\mathbf{V}$ (and, of course, are zero outside of $\mathbb{V}$) and, therefore, we can write

$$\kappa(\mathbf{r}, \tau) = \kappa(\tau) , \quad \kappa(\mathbf{r}, \omega) = \kappa(\omega) , \quad \mathbf{r} \in \mathbb{V} . \tag{5.101}$$

The assumption (ii) implies that we can approximate the incident field in $\mathbb{V}$ by a spatially-independent time-varying vector, viz,

$$\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \mathbf{E}_{\text{inc}}(t) , \quad \mathbf{r} \in \mathbb{V} . \tag{5.102}$$

In (5.102), the Cartesian components of $\mathbf{E}_{\text{inc}}(t)$ are stationary but otherwise arbitrary functions of time. Note that it does not follow from (5.102) that the incident field in $\mathbb{V}$ is equivalent to a field of a plane wave. The equivalence always holds for strictly monochromatic fields but not more generally. In the polychromatic case, (5.102) coincides with a field of a plane wave only if one of the Cartesian components of $\mathbf{E}_{\text{inc}}(t)$ is zero. Otherwise, the tip of the vector $\mathbf{E}_{\text{inc}}(t)$ is not constrained to a plane and moves quasi-chaotically in three dimensions. Consequently, polarization of $\mathbf{E}_{\text{inc}}(t)$ may not be conventionally defined in terms of the Stokes parameters. An example of this somewhat counter-intuitive behavior is illumination by two

or more non-monochromatic plane waves propagating in different directions. This observation illustrates a subtle difference that exists in applying the quasi-static approximation to monochromatic and polychromatic fields.

5.9.2 Dipole polarizability

In the considered approximations, we can fully characterize the target by a contraction of the T-matrix known as the dipole polarizability tensor, denoted by $\alpha(\tau)$ in time domain or $\alpha(\omega)$ in frequency domain (the two functions are, of course, Fourier transforms of each other). We define the total dipole moment of the target as

$$\mathbf{d}(t) = \int_{\mathbb{V}} \mathbf{P}(\mathbf{r}, t) \mathrm{d}^3 r \; . \tag{5.103}$$

Upon using the expression (5.10b) for $\mathbf{P}(\mathbf{r}, t)$, integration over $\mathbf{r}$, and using the homogeneous approximation (5.102) for $\mathbf{E}_{\mathrm{inc}}(\mathbf{r}, t)$, we obtain

$$\mathbf{d}(t) = \int_0^\infty \alpha(\tau) \mathbf{E}_{\mathrm{inc}}(t - \tau) \mathrm{d}\tau \; , \tag{5.104}$$

where

$$\alpha(\tau) = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \; \mathsf{T}_0(\mathbf{r}, \mathbf{r}'; \tau) \; . \tag{5.105a}$$

As all functions of τ above have Fourier transforms, we can also define the polarizability tensor in frequency domain as

$$\alpha(\omega) = \int_{\mathbb{V}} \mathrm{d}^3 r \int_{\mathbb{V}} \mathrm{d}^3 r' \; \mathsf{T}_0(\mathbf{r}, \mathbf{r}'; \omega) \; . \tag{5.105b}$$

Equations (5.105) define the dipole polarizability tensor in terms of the quasi-static approximation to the T-matrix, T_0 [20]. Here T_0 still satisfies the integral equation (5.20) but with the Green's tensor G_R replaced by G_0 (defined in (4.13b)). To see that G_0 is the quasi-static limit of G_R, we can take the formal limit $c \to \infty$ of (5.12b). In time domain, the integral equation for T_0 simplifies to

---

[20] In principle, we can define the dipole moment and the linear coefficient between the dipole moment and the incident field at some point in space for any target. Theories that use dipole polarizabilities outside of the quasi-static approximation are known as extended [66, 67]. Here we stay strictly within the quasi-static approximation and therefore use T_0 in place of T.

$$\mathsf{T}_0(\mathbf{r}, \mathbf{r}'; \tau) = \kappa(\tau)\delta(\mathbf{r} - \mathbf{r}')\mathtt{I}_3 + \int_0^\infty \kappa(\eta)\mathrm{d}\eta \int_{\mathbb{V}} \mathrm{d}^3 r'' \ \mathsf{G}_0(\mathbf{r}, \mathbf{r}'')\mathsf{T}_0(\mathbf{r}'', \mathbf{r}'; \tau - \eta) \ ,$$

$$(5.106)$$

where we have used the spatial uniformity condition (5.101). In frequency domain, this equation is of an even simpler form:

$$\mathsf{T}_0(\mathbf{r}, \mathbf{r}'; \omega) = \kappa(\omega)\delta(\mathbf{r} - \mathbf{r}')\mathtt{I}_3 + \kappa(\omega) \int_{\mathbb{V}} \mathrm{d}^3 r'' \ \mathsf{G}_0(\mathbf{r}, \mathbf{r}'')\mathsf{T}_0(\mathbf{r}'', \mathbf{r}'; \omega) \ . \quad (5.107)$$

This equation can be discretized (i.e., using rectangular volume elements) and the resulting system of algebraic equations can be solved to obtain a numerical approximation to the T-matrix. This approach forms the mathematical basis of the DDA [32–35]. In the quasi-static approximation, the DDA is especially efficient as the computationally-intensive part of the algorithm is independent of frequency. This is true even if the discrete representation of the T-matrix is too large to be computed or diagonalized directly; in quasistatics, there exist iterative spectral methods that are independent of the frequency or the dielectric permittivity of the target yet employ only matrix-vector multiplications. This includes methods based on the continued fraction expansion of the resolvent [68].

We can also obtain a convenient spectral representation of the polarizability tensor. This can be done in several different ways, in particular, by utilizing eigen-decomposition of the scattering operator $\Sigma(\omega)$ (defined in (4.83)). Here we follow a less formal approach and start with the integral equation for the field of electric polarization, $\mathbf{P}(\mathbf{r}, t)$. In time domain, and within the adopted approximations, this equation has the following form:

$$\mathbf{P}(\mathbf{r}, t) = \int_0^\infty \mathrm{d}\tau \ \kappa(\tau) \ \mathbf{E}_{\mathtt{inc}}(t - \tau) + \int_0^\infty \mathrm{d}\tau \ \kappa(\tau) \int_{\mathbb{V}} \mathrm{d}^3 r' \ \mathsf{G}_0(\mathbf{r}, \mathbf{r}')\mathbf{P}(\mathbf{r}', \ t - \tau) \ .$$

$$(5.108)$$

An interesting feature of this equation is that the integral kernel $\mathsf{G}_0(\mathbf{r}, \mathbf{r}')$ is real, symmetric and time-independent, and therefore its eigenfunctions $\mathbf{P}_n(\mathbf{r})$ form a complete, orthonormal, time-independent basis in $\mathbb{V}$. We denote the real eigenvalues corresponding to $\mathbf{P}_n(\mathbf{r})$ by $1/\kappa_n$ (this notation will become clear momentarily) [21]. The eigenfunctions and eigenvalues satisfy the equation

---

[21] We have labeled the eigenfunctions and eigenvalues by the discrete index n. However, if the target has a non-differentiable surface, the spectrum of eigenvalues can be continuous either globally or in some interval. While this fact is not conceptually important, it can lead to slow numerical convergence, especially, for highly-conducting metallic targets. Physically, this corresponds to the known effect of very large electric fields induced near sharp, highly-conducting features such as needles or sharp edges.

$$\int_{\mathbb{V}} \mathsf{G}_0(\mathbf{r}, \mathbf{r}') \mathbf{P}_n(\mathbf{r}') d^3 r' = \frac{1}{\kappa_n} \mathbf{P}_n(\mathbf{r}) , \quad \mathbf{r} \in \mathbb{V} , \tag{5.109}$$

and the orthonormality condition is of the form

$$\int_{\mathbb{V}} \mathbf{P}_n(\mathbf{r}) \cdot \mathbf{P}_m(\mathbf{r}) \, d^3 r = \delta_{nm} . \tag{5.110}$$

The fact that the eigenvalues and eigenfunctions are real and time- and -frequency-independent is a useful feature of the quasi-static approximation. In particular, $\mathbf{P}_n(\mathbf{r})$ and $1/\kappa_n$ are determined only by the shape of the target but not by its material properties.

Relying on the completeness of the basis of $\mathbf{P}_n(\mathbf{r})$, we can seek the solution to (5.108) in the form

$$\mathbf{P}(\mathbf{r}, t) = \sum_n \mathbf{P}_n(\mathbf{r}) g_n(t) , \tag{5.111}$$

where the time functions $g_n(t)$ are to be determined from (5.108). We can use the expansion (5.111) to write the total dipole moment of the target as

$$\mathbf{d}(t) = \sum_n \mathbf{D}_n \, g_n(t) , \tag{5.112}$$

where

$$\mathbf{D}_n = \int_{\mathbb{V}} \mathbf{P}_n(\mathbf{r}) d^3 r \tag{5.113}$$

is the dipole moment of the n-th mode. Note the relations

$$\sum_n \mathbf{D}_n \otimes \mathbf{D}_n = V[\mathbb{V}] \, \mathsf{I}_3 , \quad \sum_n \mathbf{D}_n \cdot \mathbf{D}_n = 3V[\mathbb{V}] , \tag{5.114}$$

where $V[\mathbb{V}]$ is the volume of $\mathbb{V}$.

To obtain an expression for the polarizability $\alpha(t)$, we need to relate the functions $g_n(t)$ to $\mathbf{E}_{\text{inc}}(t)$. By direct substitution of (5.111) into (5.108) and using the orthogonality of $\mathbf{P}_n(\mathbf{r})$, we find the equation for $g_n(t)$:

$$g_n(t) = \int_0^\infty \kappa(\tau) [\mathbf{D}_n \cdot \mathbf{E}_{\text{inc}}(t - \tau)] d\tau \, + \, \frac{1}{\kappa_n} \int_0^\infty \kappa(\tau) g_n(t - \tau) d\tau . \tag{5.115a}$$

We recognize that neither $g_n(t)$ nor $\mathbf{E}_{\text{inc}}(t)$ have temporal Fourier transforms. Nevertheless, from general linearity and causality, we can find the following solution to (5.115a):

$$g_n(t) = \int_0^\infty \mathbf{b}_n(\tau) \cdot \mathbf{E}_{\text{inc}}(t - \tau) d\tau , \qquad (5.115b)$$

where

$$\mathbf{b}_n(\tau) = \int_{-\infty}^\infty \frac{\mathbf{D}_n e^{-\mathrm{i}\,\omega\tau}}{1/\kappa(\omega) - 1/\kappa_n} \frac{d\omega}{2\pi} . \qquad (5.115c)$$

For most target materials that are encountered in practice, $\mathbf{b}(\tau)$ is an exponentially decreasing function of τ. Therefore, the integral transform (5.115b) is well defined and gives the general solution to (5.115a) even if $\mathbf{E}_{\text{inc}}(t)$ is not Fourier-transformable. This can be verified by direct substitution. Note that $\mathbf{b}_n(\tau)$ are causal, which means that the integral in the right-hand side of (5.115c) evaluates to zero for $\tau < 0$. This follows from the analytical properties of $\kappa(\omega)$ and the fact that all κ_n are real. Consequently, $\mathbf{b}_n(\tau) = 0$ for $\tau < 0$, and we can expand time integration in (5.115b) to the entire real axis.

Now it should be clear why we denoted the eigenvalues of $\mathsf{G}_0(\mathbf{r}, \mathbf{r}')$ by $1/\kappa_n$: the constants κ_n are special values of the function $\kappa(\omega)$. The integrand in (5.115c) has poles at the complex frequencies $\tilde{\omega}_\mu$ satisfying $\kappa(\tilde{\omega}) = \kappa_n$. These poles are known as the natural frequencies of the target [22]. In the absence of an external field, the dipole moment of the target is a superposition of oscillations of the form $\mathbf{d}_\mu \exp(-\mathrm{i}\,\tilde{\omega}_\mu t)$. We therefore expect on physical grounds that all natural frequencies lie in the lower half of the complex plane so that $\text{Im}\tilde{\omega}_\mu < 0$, with the inequality being strict. The inequality in fact holds for all causal and passive materials.

Finally, combining (5.115b), (5.112) and (5.104), we find the expression for the polarizability tensor in terms of the eigenvalues $1/\kappa_n$ and the mode dipole moments $\mathbf{D}_n$:

$$\alpha(\omega) = \sum_n \frac{\mathbf{D}_n \otimes \mathbf{D}_n}{1/\kappa(\omega) - 1/\kappa_n} . \qquad (5.116)$$

This equation provides a convenient recipe to compute $\alpha(\omega)$ numerically. Usually, the frequency-domain function $\kappa(\omega)$ is known as it can be directly related to the dielectric permittivity of the target according to (4.18). Therefore, it is more common to perform such computations in frequency domain. There exist many different ways to do so. One can use (5.116) directly, or solve numerically the Laplace equation, solve the integral equation (5.108) or its algebraic approximation, etc. Note that spectral methods, which rely on computing the modes $\mathbf{P}_n(\mathbf{r})$ and their integrals $\mathbf{D}_n$ [26, 68] are especially powerful as they allow one to compute $\alpha(\omega)$ as an analytical function of ω. The computationally-intensive part of the spectral methods depends only on the target shape but not on its material or frequency. Once $\alpha(\omega)$ is computed, the time-domain polarizability $\alpha(\tau)$ can be obtained by Fourier transform. Although, it is not possible to compute analytically the Fourier integral in (5.116) generically,

---

[22] The concept of a natural frequency can be generalized beyond quasi-statics.

in many special cases this can be done. A prime example are media with Lorentz dispersion whose analytical properties are summarized in Sec. 7.1. In this case,

$$\alpha(\tau) = \Theta(\tau) \sum_n \frac{\mathbf{D}_n \otimes \mathbf{D}_n}{\tau_n} e^{-(\gamma/2)\tau} \sin\left(\tau\sqrt{\omega_n^2 - (\gamma/2)^2}\right) . \qquad (5.117)$$

Here τ_n are positive coefficients of the dimensionality of time, ω_n's are positive frequencies and γ is a positive relaxation constant. Specific expressions for τ_n and ω_n in terms of the parameters of the Lorentz model are given in Sec. 7.1.

5.9.3 Extinction

We now turn to the extinguished power. We start from the expression (5.22) and account for the assumption that the incident field is independent of position in $\mathbb{V}$, as stated in (5.102). This results in

$$Q_{\text{ext}} = \left\langle \frac{\partial \mathbf{d}(t)}{\partial t} \cdot \mathbf{E}_{\text{inc}}(t) \right\rangle . \qquad (5.118)$$

We then utilize the time-domain linear relation (5.104) between $\mathbf{d}(t)$ and $\mathbf{E}_{\text{inc}}(t)$ and obtain

$$Q_{\text{ext}} = \left\langle \left[\frac{\partial}{\partial t} \int_0^\infty \alpha(\tau)\mathbf{E}_{\text{inc}}(t - \tau)\mathrm{d}\tau \right] \cdot \mathbf{E}_{\text{inc}}(t) \right\rangle . \qquad (5.119)$$

As was done frequently in Sec. 5.4, we utilize the identity $\partial f(t - \tau)/\partial t = -\partial f(t - \tau)/\partial \tau$, and, writing expanding all matrix-vector and dot products, obtain

$$Q_{\text{ext}} = -\sum_{ij} \int_0^\infty \alpha_{ij}(\tau)\frac{\partial}{\partial \tau} \left\langle [\mathbf{E}_{\text{inc}}(t - \tau)]_j \, [\mathbf{E}_{\text{inc}}(t)]_i \right\rangle \mathrm{d}\tau . \qquad (5.120)$$

The time averages in (5.120) are the elements of the coherence matrix for the incident field, $\Gamma_{\text{inc}}(\tau)$. Note that, in the quasi-static approximation, the incident coherence matrix is position-independent as long as $\mathbf{r}, \mathbf{r}' \in \mathbb{V}$. It can be said that the incident field is spatially coherent over $\mathbb{V}$. We therefore can re-write (5.120) in terms of the coherence matrix as

$$Q_{\text{ext}} = -\text{Tr} \int_0^\infty \alpha(\tau)\frac{\partial}{\partial \tau}\Gamma_{\text{inc}}(\tau)\mathrm{d}\tau . \qquad (5.121)$$

This expression is similar to (5.26) except that the spatial integrals have now been reduced to the single parameter $\alpha(\tau)$. We can further make the same steps as in Sec. 5.4 and transform (5.121) into frequency domain, viz,

$$Q_{\text{ext}} = \frac{1}{\pi}\text{Tr Im} \int_0^\infty \mathtt{W}_{\text{inc}}(\omega) \; \alpha(\omega) \; \omega \text{d}\omega \; . \tag{5.122}$$

Since $\alpha(\omega)$ is symmetric and $\mathtt{W}(\omega)$ is Hermitian, we also have

$$Q_{\text{ext}} = \frac{1}{\pi}\text{Tr} \int_0^\infty \mathtt{W}_{\text{inc}}^{(\text{r})}(\omega) \; \alpha^{(\text{i})}(\omega) \; \omega \text{d}\omega \; , \tag{5.123}$$

which is similar to the more general result (5.43), which was derived above.

(a) Monochromatic field

If the incident field (5.102) is monochromatic, it is indistinguishable in $\mathbb{V}$ from the field of a monochromatic plane wave. Let $\mathbf{E}_{\text{inc}}(t) = \text{Re}[\boldsymbol{A}e^{-\text{i}\,\omega_0 t}]$, where $\boldsymbol{A}$ is a complex amplitude. Then the field is polarized in the plane spanned by the vectors $(\boldsymbol{A}^{(\text{r})}, \boldsymbol{A}^{(\text{i})})$. The propagation direction is perpendicular to this plane but otherwise undetermined. This indicates that the extinguished power for the considered setup (quasi-static approximation, monochromatic incident plane wave) is invariant with respect to reversing the propagation direction while keeping the polarization plane fixed. This is not equivalent to saying that the extinguished power is independent of the propagation direction. Indeed, changing the propagation direction entails changing the plane of polarization (since the incident wave is transverse), and extinction is obviously sensitive to polarization.

For the incident field introduced above, we have

$$[\mathtt{W}_{\text{inc}}(\omega)]_{ij} = \frac{\pi}{2}\left[A_i^* A_j \delta(\omega - \omega_0) + A_i A_j^* \delta(\omega + \omega_0)\right] \; , \tag{5.124}$$

and, therefore,

$$Q_{\text{ext}} = \frac{\omega_0}{2}\text{Im}[\boldsymbol{A}^* \cdot \alpha(\omega_0)\boldsymbol{A}] = \frac{\omega_0}{2}\left[\boldsymbol{A}^* \cdot \alpha^{(\text{i})}(\omega_0)\boldsymbol{A}\right] \; . \tag{5.125}$$

Note that, even though $\boldsymbol{A}$ can be complex, the second expression in (5.125) is real because $\alpha^{(\text{i})}(\omega)$ is a real symmetric and hence Hermitian tensor. For this reason, we have omitted the real part symbol in this formula.

Equation (5.125) is a well-known result for the extinguished power in the dipole approximation wherein the electromagnetic properties of a target are reduced to its dipole polarizability tensor $\alpha(\omega)$. As mentioned above, monochromatic incident field is indistinguishable in $\mathbb{V}$ from the field of some monochromatic plane wave. Therefore, Eq. (5.125) applies to a monochromatic plane wave propagating in the direction perpendicular to the plane spanned by the two real vectors $(\boldsymbol{A}^{(\text{r})}, \boldsymbol{A}^{(\text{i})})$. One conclusion that we can make is that there is no chirality in the quasi-static approximation. More precisely, the target may have a chiral shape geometrically, but this has no effect on extinction. This follows from the invariance of (5.125) with

respect to the substitution $A \to A^*$. This is consistent with the observation made above that quasi-static extinction due to a monochromatic plane wave is invariant with respect to reversing the direction of propagation while keeping the polarization plane fixed.

(b) Quasi-monochromatic plane wave

Let the incident field be of the form $\mathbf{E}_{\text{inc}}(t) = \text{Re}[A(t)e^{-i\omega_0 t}]$, where $A(t)$ is a slow-varying complex amplitude. In general, this field does not have a fixed plane of polarization. We therefore make the additional assumption that $A(t)$ is constrained to the XY-plane at any moment of time. In this case, $\mathbf{E}_{\text{inc}}(t)$ is indistinguishable in $\mathbb{V}$ from the field of some quasi-monochromatic plane wave propagating in either positive or negative Z-direction. The cross-spectral density for a general quasi-monochromatic plane wave was derived in Sec. 5.8.2 above and is given in (5.95). In the quasi-static approximation, the factor $e^{i(z'-z)/c}$ in this formula can be replaced by unity. Expressing the time-averages in (5.95) in terms of the Stokes parameters according to (5.97)

$$\mathsf{W}_{\text{inc}}(z, z'; \omega) = 2\pi \begin{bmatrix} I + Q & U + iV \\ U - iV & I - Q \end{bmatrix} \delta(\omega - \omega_0) . \qquad (5.126)$$

Upon substitution of this formula into (5.122), and accounting for the symmetry $\alpha_{xy}(\omega) = \alpha_{yx}(\omega)$, we obtain

$$Q_{\text{ext}} = 2\omega_0 \left\{ I \left[\alpha_{xx}^{(i)}(\omega_0) + \alpha_{yy}^{(i)}(\omega_0)\right] \right.$$
$$\left. + Q \left[\alpha_{xx}^{(i)}(\omega_0) - \alpha_{yy}^{(i)}(\omega_0)\right] + 2U \, \alpha_{xy}^{(i)}(\omega_0) \right\} . \qquad (5.127)$$

As expected, the Stokes parameter V does not enter the above expression, which confirms our previous conclusion that there is no electromagnetic manifestation of the target chirality within the quasi-static approximation. Moreover, since $\alpha^{(i)}(\omega_0)$ is a real symmetric 2×2 tensor, we can always rotate the X and Y axes about the Z axis so that $\alpha_{xy}^{(i)}(\omega_0) = 0$ [23]. Correspondingly, we may write

$$Q_{\text{ext}} = 2\omega_0 \left\{ I \left[\alpha_{xx}^{(i)}(\omega_0) + \alpha_{yy}^{(i)}(\omega_0)\right] + Q \left[\alpha_{xx}^{(i)}(\omega_0) - \alpha_{yy}^{(i)}(\omega_0)\right] \right\} . \qquad (5.128)$$

$$\text{(in principal axes)}$$

In this case, the parameter U is also unimportant.

Further, we can write

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[23] It might not be possible to find axes in which the *complex* matrix element satisfies $\alpha_{xy}(\omega_0) = 0$, and the axes in which $\alpha_{xy}^{(i)}(\omega_0) = 0$ may depend on the frequency.

$$\alpha^{(i)}(\omega_0) = \sum_{i=1}^{3} \alpha_i \, \mathbf{u}_i \otimes \mathbf{u}_i \, , \qquad (5.129)$$

where α_i $(i = 1, 2, 3)$ is a set of real principal values of the 3×3 real symmetric tensor $\alpha^{(i)}(\omega_0)$ and $\mathbf{u}_i$ are the corresponding orthogonal principal vectors satisfying $\mathbf{u}_i \cdot \mathbf{u}_j = \delta_{ij}$ (all quantities can depend on ω_0). It follows that the orientation average of the extinguished power can depend only on the principal values α_i and, moreover, all three principal values must enter the averaged quantity on equal footing. By orientation averaging we mean here average over such rotations of the target that, say, the first of the principal vectors $\mathbf{u}_1$ samples uniformly the unit sphere and the triplet $(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ is rotated as a rigid body. Putting all these symmetry arguments together, we conclude that the orientation-averaged power of extinction is

$$\langle Q_{\text{ext}} \rangle_{\text{orientation}} = \frac{4\pi}{3} \omega_0 \, I \, \text{Tr}[\alpha^{(i)}(\omega_0)] \, . \qquad (5.130)$$

Thus, only the overall intensity of the incident field, I, is important for the orientation average. A more rigorous mathematical analysis of orientation averaging of extinction can be found in [69].

6 Operational definition and paradoxes

6.1 Motivation and review

We have discussed so far how to compute the extinguished, absorbed and scattered powers theoretically given the incident field and the target. We however did not yet consider the question how any of these powers can be *measured*. In this section, we focus on measuring the extinguished power for incident plane waves as this is the commonly-encountered experimental setup either for natural or artificial light sources. Scattered and absorbed powers can also be measured, but this typically requires wide area or wide solid angle measurements. In contrast, extinction is special because, by optical theorem, it is related to the scattering amplitude in the forward direction (Sec. 4.4). It can therefore be hoped that extinction can be measured with a relatively small, flat, power-integrating detector placed on axis with the target. While this conjecture is, basically, correct, the problem of measuring extinction is non-trivial and attracted significant attention in the past decade or so [70–74]. One of the defining contributions was made by Mishchenko *et al.* in 2009 [16] and a recent review of the field can be found in [75].

Undoubtedly, the reason why the questions considered in the above references were not settled a long time ago is related to the counter-intuitive nature of extinction. The extinguished energy current is not directly measurable, and it is fundamentally impossible to spatially separate the incident and scattered fields (there would be no extinction otherwise). The same reasons have led to the so-called extinction

paradoxes, which challenge our intuition of what extinction is or should be. Therefore, in the end of this section, we will discuss two extinction paradoxes, including the classical paradox of extinction cross section being twice as large as the geometrical cross section for electrically large targets.

We note that, in many cases, measuring extinction is sufficient to know absorption and scattering. For example, macroscopically large water droplets in the atmosphere have very low absorption. In this case, extinguished and scattered powers (or cross sections) are almost the same. At the other extreme, small metal colloidal particles in solutions can often be treated within the quasi-static approximation according to which scattering is negligible compared to absorption. In this case, extinction and absorption are almost the same.

6.2 The measurement planes

The first question we wish to address is whether there is a well-defined surface on which the extinction can be measured with a flat power-integrating detector. To this end, it is instructive to introduce two measurement planes denoted by $\mathbb{S}_{\text{front}}$ and $\mathbb{S}_{\text{back}}$ as is illustrated in Fig. 3. The surface $\mathbb{S}_{\text{front}}$ is located between the source and the target, and every ray that connects the two objects must cross this plane. The plane $\mathbb{S}_{\text{back}}$ is located behind the target. We assume that the two planes are parallel. We also show in the figure two spatial regions Ω and Ω'. The region Ω contains the target and has the same properties as the region Ω that was defined in Sec. 3.1. In Fig. 1, this region is shown as spherical and in Fig. 3 it is deformed so that its surface overlaps with $\mathbb{S}_{\text{front}}$ and $\mathbb{S}_{\text{back}}$. However, this difference does not affect the definition and properties of Ω. We can imagine that Ω is a truncated cylinder or a cuboid with the bases located in $\mathbb{S}_{\text{front}}$ and $\mathbb{S}_{\text{back}}$ and the height H equal to the distance between the two planes. The region Ω' is different: it contains the source but no part of the target. The boundaries $\partial\Omega$ and $\partial\Omega'$ share the same sub-region of $\mathbb{S}_{\text{front}}$, which, as suggested above, can be a circle or a rectangle. Importantly, this sub-region must be large enough to encompass a dominant part (over the boundary $\partial\Omega'$) of the interaction between the incident field and the field scattered by the target. We will show that the total flux of $\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r})$ (defined in (3.12)) through $\partial\Omega'$ is zero and, by extension, the total flux of $\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r})$ through $\mathbb{S}_{\text{front}}$ is also zero. Consequently, the extinguished power Q_{ext}, defined as the surface integral (3.11), enters the three-dimensional region Ω entirely through $\mathbb{S}_{\text{back}}$. This conclusion may seem to be counter-intuitive, but it is consistent with the physical understanding of extinction as removal of energy from a beam of radiation. We will however need to consider some mathematical nuances to see how this interpretation works in practice.

The first statement, namely, that

$$\oint_{\partial\Omega'} \overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r})\, \mathrm{d}^2 r = 0 \tag{6.1}$$
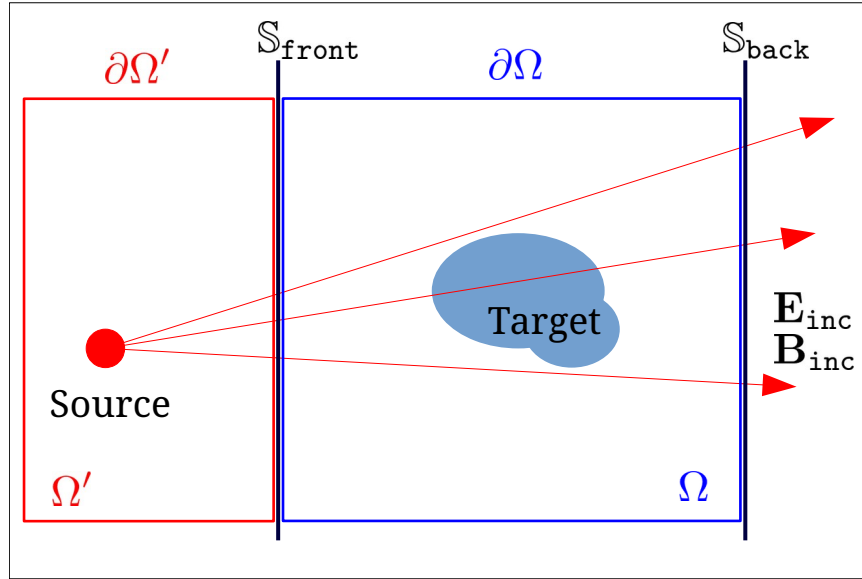
Fig. 3 Schematic illustration of the measurement planes $\mathbb{S}_{\texttt{front}}$ and $\mathbb{S}_{\texttt{back}}$. The total extinction flux through $\mathbb{S}_{\texttt{front}}$ is zero.

is easy to prove. Indeed, we have for the energy current on $\partial\Omega'$

$$\overline{\mathbf{S}}(\mathbf{r}) = \overline{\mathbf{S}}_{\texttt{inc}}(\mathbf{r}) + \overline{\mathbf{S}}_{\texttt{sca}}(\mathbf{r}) + \overline{\mathbf{S}}_{\texttt{ext}}(\mathbf{r}) \,, \tag{6.2}$$

where $\overline{\mathbf{S}}_{\texttt{sca}}$ and $\overline{\mathbf{S}}_{\texttt{ext}}$ are defined in (3.9) and (3.12), and we assume a similar definition for $\overline{\mathbf{S}}_{\texttt{inc}}$ (involving only the incident field). Since Ω' contains the source but not the target, the total outward energy flux through $\partial\Omega'$, computed as a surface integral of $\overline{\mathbf{S}}(\mathbf{r})$, is equal to the power of the source, $Q_{\texttt{inc}}$. The total power flux of the incident field through $\partial\Omega'$, computed as a surface integral of $\overline{\mathbf{S}}_{\texttt{inc}}(\mathbf{r})$, is also equal to $Q_{\texttt{inc}}$. This follows from the assumption of no back-action (stated in Sec. 3.1), according to which the power of the source is independent of presence or absence of the target. Consequently, integrating (6.2) over $\partial\Omega'$, we obtain

$$\oint_{\partial\Omega'} \overline{\mathbf{S}}_{\texttt{sca}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, \mathrm{d}^2 r + \oint_{\partial\Omega'} \overline{\mathbf{S}}_{\texttt{ext}}(\mathbf{r}) \cdot \mathbf{n}(\mathbf{r}) \, \mathrm{d}^2 r = 0 \,. \tag{6.3}$$

The integral of $\overline{\mathbf{S}}_{\texttt{sca}}(\mathbf{r})$ is obviously zero (since the "secondary source" of $\overline{\mathbf{S}}_{\texttt{sca}}(\mathbf{r})$ is supported outside of Ω') and we have therefore proved (6.1). The proof is mathematically rigorous and not an approximation, as long as we neglect the back-action of the scatterer on the source. We have stated the absence of back-action as a fundamental assumption of the theory in Sec. 3.1. However, we should keep in mind that the scattered field is not strictly zero in the region of space where the source

currents are supported. In some problems where the source is small (i.e., a quantum emitter such as a molecule) and the target is large (i.e., a macroscopic substrate), the back-action produces the Purcell effect, and the corresponding Purcell factor can be viewed as a quantitative measure of the interaction. An example of a purely classical computation of the Purcell factor has been given in [28], and a more general theory of energy budgets in a setting where the source and the target can interact has been developed in [76]. In a typical scattering problem, the source and the target are placed sufficiently far apart, so that the back-action power can be neglected not only when compared to the total power of the source but also relative to the power extinguished by the target. We proceed under this assumption.

In order to prove the stronger statement that

$$\oint_{\mathbb{S}_{\mathrm{front}}} \overline{\mathbf{S}}_{\mathrm{ext}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r = 0 \,, \tag{6.4}$$

we need an additional assumption that the incident field is in some sense forward-directed and therefore interference of the incident field and the field scattered by the target occurs on $\mathbb{S}_{\mathrm{front}}$ but not on the rest of $\partial\Omega'$. In problems involving collimated beam or plane-wave illumination, this assumption holds naturally. In other cases, we can come to the same conclusion by sending the size of overlap of $\partial\Omega'$ and $\mathbb{S}_{\mathrm{front}}$ to infinity.

We thus arrived at an important conclusion: the extinguished power is equal to the total flux of $\overline{\mathbf{S}}_{\mathrm{ext}}(\mathbf{r})$ through $\mathbb{S}_{\mathrm{back}}$. Mathematically, the statement is formalized as

$$Q_{\mathrm{ext}} = - \int_{\mathbb{S}_{\mathrm{back}}} \overline{\mathbf{S}}_{\mathrm{ext}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r \,. \tag{6.5}$$

We emphasize that (6.5) was not proved rigorously but only made plausible. The conclusion relies on the assumption that interference of the incident and scattered fields is negligible on the parts of $\partial\Omega'$ that do not overlap with $\mathbb{S}_{\mathrm{front}}$. For most physical settings in which measuring extinction is of practical interest, the overlap of $\partial\Omega'$ and $\mathbb{S}_{\mathrm{front}}$ can be made sufficiently large for this assumption to hold with arbitrary precision. However, one can think of exotic cases for which satisfying the condition may be challenging.

6.3 Measuring extinction

Equation (6.5) still contains a quantity that is not directly measurable. Indeed, $\overline{\mathbf{S}}_{\mathrm{ext}}(\mathbf{r})$, as defined in (3.12), is a peculiar interference term, which involves a quadratic combination of the incident and scattered fields. The measurable fields on $\mathbb{S}_{\mathrm{back}}$ are the *total* fields $\mathbf{E}(\mathbf{r}, t)$ and $\mathbf{B}(\mathbf{r}, t)$. However, we can still access Q_{ext} as

a differential measurement, and this will reveal the physical significance of extinction.

Consider the two measurements schematically illustrated in Fig. 4. Here we assume illumination by a plane wave propagating along the Z-axis. The plane $\mathbb{S}_{\text{back}}$ is perpendicular to Z. A flat power-integrating detector measures the total energy flux intercepted by its surface, which we denote by $\mathbb{S}_{\text{det}}$. The measured power can be written as

$$Q_{\text{det}} = \int_{\mathbb{S}_{\text{det}}} \overline{\mathbf{S}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r \,, \tag{6.6}$$

where $\mathbf{e}_z$ is the unit vector pointing in the Z-direction. Assume that we can perform this measurement with and without the target [24] and let the respective measurements be $Q_{\text{det}}^{(a)}$ and $Q_{\text{det}}^{(b)}$. Then

$$\Delta Q_{\text{det}} = Q_{\text{det}}^{(b)} - Q_{\text{det}}^{(a)} = \int_{\mathbb{S}_{\text{det}}} \left[\overline{\mathbf{S}}_{\text{inc}}(\mathbf{r}) - \left(\overline{\mathbf{S}}_{\text{inc}}(\mathbf{r}) + \overline{\mathbf{S}}_{\text{sca}}(\mathbf{r}) + \overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \right) \right] \cdot \mathbf{e}_z \, \mathrm{d}^2 r$$

$$= -\int_{\mathbb{S}_{\text{det}}} \overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r - \int_{\mathbb{S}_{\text{det}}} \overline{\mathbf{S}}_{\text{sca}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r \,. \tag{6.7}$$

From (6.5), we know that

$$\lim_{\text{area}[\mathbb{S}_{\text{det}}] \to \infty} \left\{ -\int_{\mathbb{S}_{\text{det}}} \overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{e}_z \, \mathrm{d}^2 r \right\} = Q_{\text{ext}} \,. \tag{6.8}$$

Therefore, a differential measurement of the type (6.7) with a sufficiently large flat detector may produce the desired quantity Q_{ext} *assuming* that the input of $\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r})$ to $Q_{\text{det}}^{(a)}$ is negligible. Therefore, the detector area needs to be large but not too large, so that the limit in (6.8) is almost reached yet the input of the scattered energy flux in (6.7) is negligible. The required intermediate value of the detector area, in fact, can be found, but only if the detector is sufficiently far from the target. This is so because the scattered field scales as $1/R$ with the distance R from the target to the detector. Consequently, $\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r})$ scales as $1/R^2$. However, $\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r})$ is an interference term with only one copy of the scattered field, and therefore it scales as $1/R$. It seems therefore that all one needs in order to make a measurement of Q_{ext} is to move the detector as far from the target as possible. This conclusion is, in principle, correct, but it has been badly muddled by the oscillatory nature of the integral in (6.8). The oscillations in question are, of course, interference fringes, which are visible only for highly coherent sources. However, since the problem is most often considered in

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[24] If the target cannot be removed, we can utilize a pair of measurements by placing the detector in front and behind the target (that is, on the planes $\mathbb{S}_{\text{front}}$ and $\mathbb{S}_{\text{back}}$). While these two measurements can be used to determine Q_{ext}, they are not exactly equivalent to the differential measurement on $\mathbb{S}_{\text{back}}$ due to a small but nonzero contributions of $\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r})$.
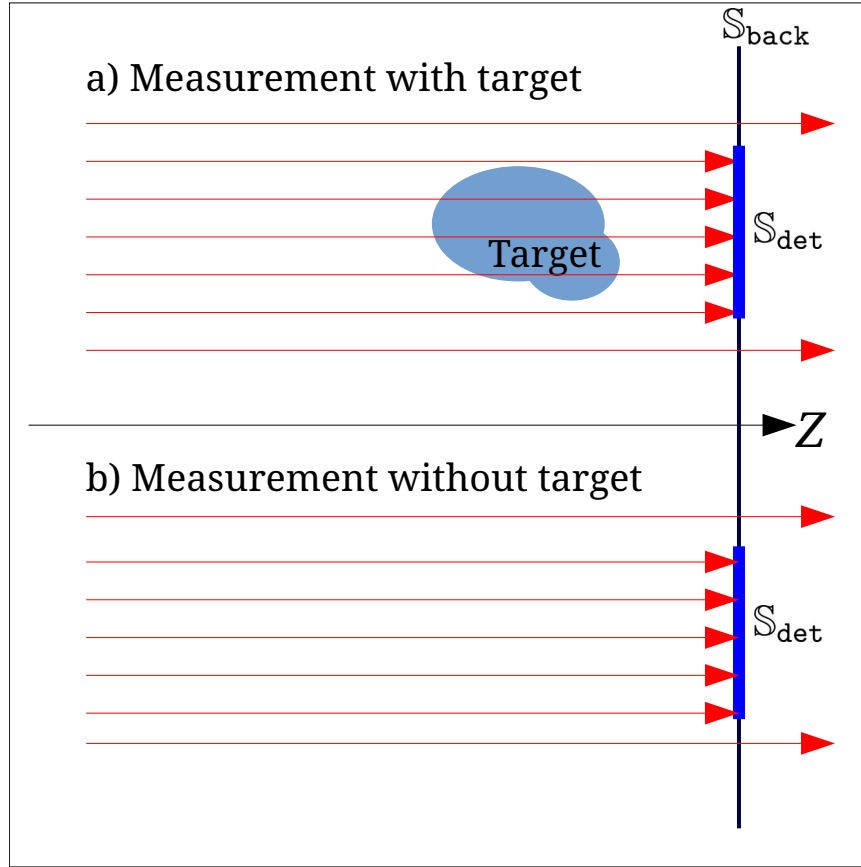
Fig. 4 Two measurements with a flat power-integrating detector of area $\mathbb{S}_{\text{det}}$. In measurement (a), the target is present and in measurement (b) the target is removed and the detector measures energy flux of the incident field.

frequency domain, it is instructive to illustrate the relevant mathematical issues with a simple physical example.

Consider the physical setup of Rayleigh scattering. Let a small particle of a scalar dipole polarizability α be placed at the origin of a reference frame XYZ and illuminated by a monochromatic, linearly-polarized incident plane wave of the form

$$\mathbf{E}_{\text{inc}}(\mathbf{r}, t) = \text{Re}[\mathbf{e}_x A e^{\text{i}(kz-\omega t)}] , \quad \mathbf{B}_{\text{inc}}(\mathbf{r}, t) = \text{Re}[\mathbf{e}_y A e^{\text{i}(kz-\omega t)}] \qquad (6.9)$$

with the associated energy current

$$\overline{\mathbf{S}}_{\text{inc}} = \frac{c}{8\pi}|A|^2 \, \mathbf{e}_z . \qquad (6.10)$$
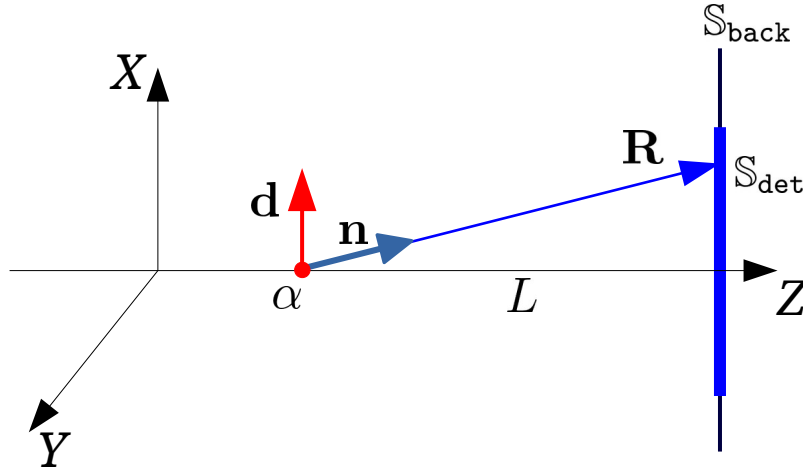
Fig. 5 Measurement geometry with a flat power-integrating detector located in the plane $\mathbb{S}_{\text{back}}$. The detector is either a circle of the radius a or a square of the side $2a$.

The dipole moment of the particle oscillates at the same frequency ω as $\mathbf{d}(t) = \text{Re}[de^{-i\omega t}]$, where $d = \alpha A \mathbf{e}_x$. The flat detector is located in the plane $\mathbb{S}_{\text{back}}$. The distance from the origin to $\mathbb{S}_{\text{back}}$ is L and the radius-vector from the origin to a point on the detector surface is $\mathbf{R} = L\mathbf{e}_z + \boldsymbol{\rho}$, where $\boldsymbol{\rho} = (X, Y)$ is a two-dimensional vector specifying a point in the plane $\mathbb{S}_{\text{back}}$. We define $\mathbf{n} = \mathbf{R}/R$ to be the unit vector pointing from the origin to the point (X, Y) on the detector. The measurement geometry is illustrated in Fig. 5.

We assume that the distance between the target and the measurement plane is large compared to the wavelength so that $kL = \omega L/c \gg 1$. In the numerical example below, we take $kL = 10^5$, so that this inequality is satisfied strongly. Under this condition, we can can keep only the far-field contribution $\mathsf{G}_2(\mathbf{r}, \mathbf{r}')$ to the Green's tensor (4.13a) and disregard $\mathsf{G}_0(\mathbf{r}, \mathbf{r}')$ and $\mathsf{G}_1(\mathbf{r}, \mathbf{r}')$. We can therefore write the complex amplitudes of the scattered fields as

$$E_{\text{sca}}(\mathbf{r}) = \frac{k^2}{R} \left[d - (d \cdot \mathbf{n})\mathbf{n} \right] e^{ikR}, \quad B_{\text{sca}}(\mathbf{r}) = \frac{k^2}{R} (\mathbf{n} \times d) \, e^{ikR}. \qquad (6.11)$$

We can now use the basic definitions of energy currents (3.9) and (3.12) and the time-averaging techniques applicable to monochromatic fields introduced in Sec. 4.1 to compute $\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r})$ and $\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r})$:

$$\overline{\mathbf{S}}_{\text{sca}}(\mathbf{r}) = S_{\text{inc}} \frac{k^4}{R^2} |\alpha|^2 \left[1 - (\mathbf{n} \cdot \mathbf{e}_x)^2 \right] \mathbf{n}, \qquad (6.12a)$$

$$\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) = S_{\text{inc}} \text{Re} \left\{ \frac{k^2}{R} \alpha e^{ik(R-z)} \left[\mathbf{n} - (\mathbf{n} \cdot \mathbf{e}_x)\mathbf{e}_x + \mathbf{e}_z - (\mathbf{n} \cdot \mathbf{e}_x)(\mathbf{n} \times \mathbf{e}_y) \right] \right\}, \qquad (6.12b)$$

where $S_{\text{inc}} = c|A|^2/8\pi$ is the scalar factor in (6.10). Integrating the fluxes of these energy currents through a spherical surface $R = R_0$ according to (3.8) and (3.11), we obtain

$$Q_{\text{sca}} = S_{\text{inc}}\sigma_{\text{sca}} \,, \quad Q_{\text{ext}} = S_{\text{inc}}\sigma_{\text{ext}} \,, \tag{6.13a}$$

where

$$\sigma_{\text{sca}} = \frac{8\pi}{3} k^4 |\alpha|^2 \,, \quad \sigma_{\text{ext}} = 4\pi k \,\text{Im}\,\alpha \,. \tag{6.13b}$$

These theoretical results are well known in Rayleigh theory of scattering. We are interested in how Q_{ext} or σ_{ext} can be measured with the flat detector shown in (5). To this end, we fix $Z = L$, project the energy currents in (6.12) onto the Z-axis, and integrate the result over the detector area $\mathbb{S}_{\text{det}}$. We will consider two cases in which $\mathbb{S}_{\text{det}}$ is a circle of the radius a and a square of the side $2a$. We then have

$$\int\limits_{\mathbb{S}_{\text{det}}} \frac{\overline{\mathbf{S}}_{\text{sca}}(X,Y,L)\cdot\mathbf{e}_z}{Q_{\text{sca}}}\,\mathrm{d}X\mathrm{d}Y = I_{\text{sca}}(a/L) \,, \tag{6.14a}$$

$$-\int\limits_{\mathbb{S}_{\text{det}}} \frac{\overline{\mathbf{S}}_{\text{ext}}(X,Y,L)\cdot\mathbf{e}_z}{Q_{\text{ext}}}\,\mathrm{d}X\mathrm{d}Y = -\frac{1}{\text{Im}\,\alpha}\,\text{Re}\left[\alpha\,I_{\text{ext}}(a/L)\right] \,, \tag{6.14b}$$

where

$$I_{\text{sca}}(\xi) = \frac{3}{8\pi} \int\limits_{\mathbb{F}[\xi]} \frac{1+y^2}{r^5(x,y)}\,\mathrm{d}x\mathrm{d}y \,, \tag{6.14c}$$

$$I_{\text{ext}}(\xi) = \frac{kL}{4\pi} \int\limits_{\mathbb{F}[\xi]} e^{ikL[r(x,y)-1]}\left[1 + \frac{1+y^2}{r(x,y)}\right]\frac{\mathrm{d}x\mathrm{d}y}{r^2(x,y)} \,, \tag{6.14d}$$

and we have introduced the dimensionless variables

$$x = \frac{X}{L} \,, \quad y = \frac{Y}{L} \,, \quad \xi = \frac{a}{L} \,, \quad r(x,y) = \sqrt{1+x^2+y^2} \,. \tag{6.14e}$$

Here $\mathbb{F}[\xi]$ is the scaled (by L) area of the detector. As mentioned above, we take $\mathbb{F}[\xi]$ to be either a circle of radius ξ or a square of side 2ξ. The quantities Q_{sca} and Q_{ext} in (6.12) are the theoretical values of the scattered and extinguished powers given in (6.13a). Note that $I_{\text{sca}}(\xi)$ is real but $I_{\text{ext}}(\xi)$ is complex. We thus see that the detector would measure (approximately) the extinguished power if there is an interval of ξ in which, simultaneously, $I_{\text{sca}}(\xi) \ll 1$ and $I_{\text{ext}}(\xi) \approx i$. Here ξ quantifies the solid angle subtended by the detector when viewed from the target.

The integral $I_{\text{sca}}(\xi)$ does not depend on kL, can be expressed in terms of elementary functions, and is illustrated graphically in Fig. 6 for both circular and square integration regions. It can be seen that, for $\xi \lesssim 0.2$, the condition $I_{\text{sca}}(\xi) \ll 1$ holds
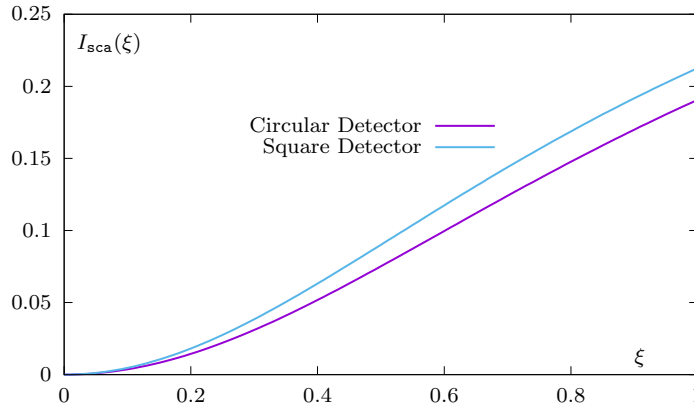
Fig. 6 Integral $I_{\mathrm{sca}}(\xi)$ for circular and square detectors. At larger values of ξ, both functions tend to 1/2. For $\xi \lesssim 0.2$, the condition $I_{\mathrm{sca}}(\xi) \ll 1$ holds with a good precision.

in both cases. Thus, we conclude that the scattered energy flux can be neglected if the detector subtends a solid angle less than, approximately, $4\pi/100$.

The integral $I_{\mathrm{ext}}(\xi)$ is considerably more complicated. It can be expressed in terms of integral trigonometric and exponential functions for a circular integration region, but, for the square, it has to be computed numerically. This is complicated due to the oscillatory integrand, especially, when kL is large. Nevertheless, we have computed $I_{\mathrm{ext}}(\xi)$ for $kL = 10^5$ as is illustrated in Fig. 7. It can be seen that, for both shapes of the detector, the integral converges to i when $\xi \to \infty$, as expected. However, in the case of a circular detector, the convergence is slow and oscillatory, and the integral is still far from its limit at $\xi = 2$. Under such conditions, measurement of Q_{ext} is impossible. However, in the case of a square detector, the oscillations are suppressed, and the integral approaches its limit already at $\xi \sim 0.1$, well in the range of ξ wherein we can neglect the contribution of $I_{\mathrm{sca}}(\xi)$. We then conclude that the measurement of Q_{ext} with a square detector is possible. At the target-detector separation of $kL = 10^5$, the relative error of such a measurement is of the order of 2%.

The observation that a square detector, unlike a circular one, can measure the extinguished power in the setup of Fig. 5 was made by Mishchenko *et al.* in 2009 [16]. However, the insight obtained in this work is even deeper. Namely, Ref. [16] points out that the oscillations seen in Fig. 7(a) occur only in a very special case of a monochromatic field and a circular detector, which is on axis with the target. Any deviation from these assumptions either destroys or integrates out the interference fringes and makes measurement of Q_{ext} possible. In particular, it was already observed in the precursors of Ref. [16] by Berg *et al.* [70, 71] that measuring Q_{ext} with a flat circular detector is easier for several scattering particles than for just one. This is so because not all particles were placed on axis with the detector. A circular
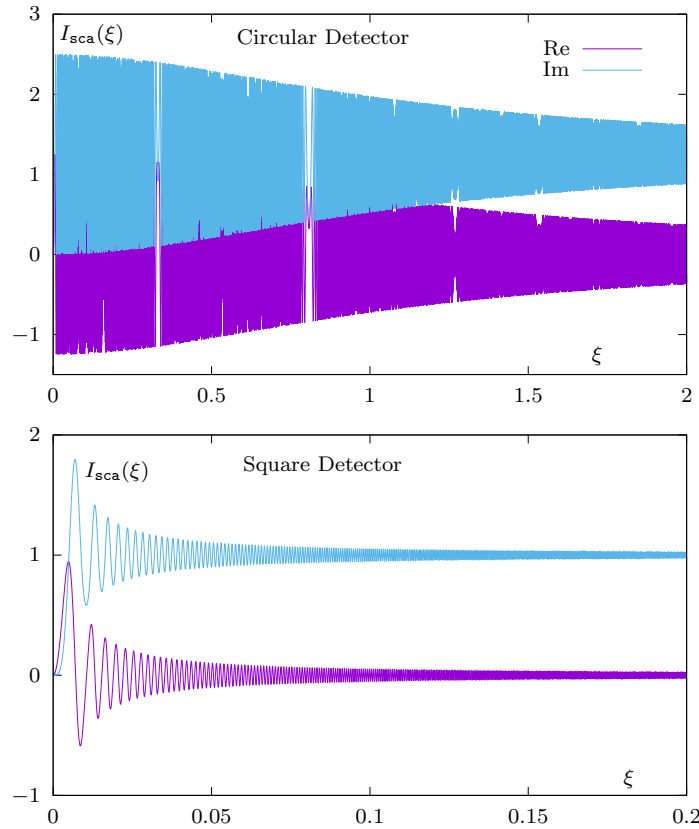
Fig. 7 Integral $I_{ext}(\xi)$ for circular (top) and square (bottom) detectors and the parameter $kL = 10^5$. In both cases, the integral approaches i when $\xi \rightarrow \infty$. However, for the square detector, approximate convergence to the limit is reached much faster.

detector that is not on axis integrates the interference fringes out similarly to an on-axis square detector.

Another important conclusion of Ref. [16] is the following. In order to measure extinction, we must make the contribution of the scattered energy flux negligible. To this end, the detector must be placed sufficiently far from the target. While we do not derive the exact condition, both the target and the detector must be deep in the Fraunhofer diffraction zone of each other. In other words, if the characteristic size of the target is D and the characteristic size of the detector is a, the distance L in Fig. 5 must satisfy $L \gg k \max(D^2, a^2)$. The inequality must be satisfied strongly since the convergence of the measurement to its theoretical limit is algebraic and therefore slow. In Fig. 7, we have taken $kL = 10^5$, which is sufficient to illustrate the effect and still feasible computationally. However, measurements are often performed at even larger values of kL. For example, if the wavelength is $\lambda = 500\text{nm}$ and $L = 1\text{m}$,
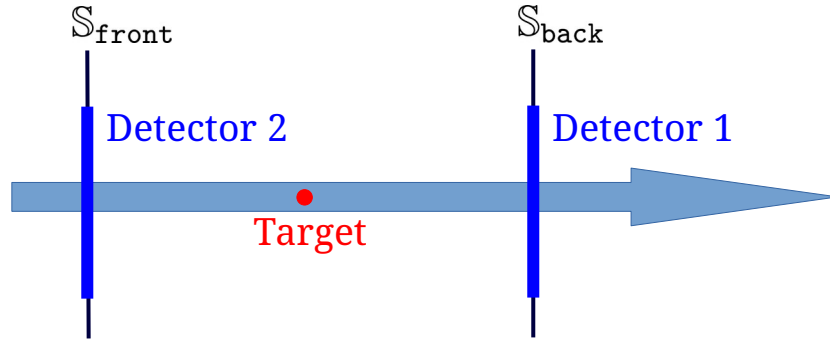
$\mathbb{S}_{\texttt{front}}$ $\mathbb{S}_{\texttt{back}}$

Detector 2   Detector 1

Target

Fig. 8 Measuring extinction in the case of a collimated incident beam. Extinguished power can be obtained as a differential measurement of Detector 1 with and without the target or as a differential measurement of powers registered by Detectors 2 and 1. When Detector 1 is used, Detector 2 should be removed from the optical path and not to block the beam. The width of the beam should be sufficiently large to fully encompass the target.

we have $kL \approx 10^7$. Although this case is difficult to simulate numerically, the actual *physical* measurement at $kL = 10^7$ is even more precise. In general, when the value of kL is increased, convergence of the differential measurement (6.7) to its theoretical limit Q_{ext} becomes faster, except in the special case of monochromatic field and a circular detector on axis with the target.

The above discussion applies to illumination by a wide-front field such as sunlight. In the case of laboratory measurements with tightly collimated beams, measuring extinction is much easier as the extinguished power is literally removed from the beam. Consider the measurement geometry illustrated in Fig. 8. Since the flux of extinguished power through $\mathbb{S}_{\text{back}}$ is zero, and the interference of the incident and scattered field on $\mathbb{S}_{\text{back}}$ occurs only on the surface of Detector 1, we can measure the extinguished power almost precisely by one of the following two differential measurements. We can measure the powers measured by Detector 1 with and without the target and take the difference (Detector 2 should be absent in this case), or, keeping the target in place, compute the difference between powers measured by Detector 2 and Detector 1 (when Detector 1 is activated, Detector 2 should be removed and not block the beam). It should be kept in mind that the beam can have a complicated structure and the measured extinction for a beam is not necessarily the same as the extinguished power for an plane wave. The correspondence is obtained if the incident beam can be approximated by a plane wave locally over $\mathbb{V}$. This is achievable with Gaussian beams of sufficiently large waist. There exists however an apparent paradox associated with the measurement scheme of Fig. 8. Extinction-related paradoxes are discussed in Sec. 6.4 below.

We finally note that the fact that $I_{\text{sca}}(\xi)$ tends to 1/2 as $\xi \to \infty$ (this result is not graphically illustrated in Fig. 6 but can be easily verified) suggests the way in which the absorbed power can be measured. Unlike Q_{ext}, Q_{abs} cannot be accessed

by placing a detector only on $\mathbb{S}_{\text{back}}$. Rather, we have

$$Q_{\text{abs}} = \int_{\mathbb{S}_{\text{front}}} \overline{\mathbf{S}}(\mathbf{r}) \cdot \mathbf{e}_z \mathrm{d}^2 r - \int_{\mathbb{S}_{\text{back}}} \overline{\mathbf{S}}(\mathbf{r}) \cdot \mathbf{e}_z \mathrm{d}^2 r . \tag{6.15}$$

Therefore, Q_{abs} can be accessed as a differential measurement with two large-area detectors. The target must be present in both measurements. For the measurement on $\mathbb{S}_{\text{back}}$, there should be no detector on $\mathbb{S}_{\text{front}}$. The areas of detectors should be large enough for the function $I_{\text{sca}}(\xi)$ to approach its theoretical limit of $1/2$, which occurs at approximately $\xi \sim 10$. Obviously, measuring absorption is a much harder task than measuring extinction.

6.4 Paradoxes

Because the extinguished power is not defined in terms of a physical energy current, it has long been associated with various paradoxes, which pitch our intuition against mathematical facts. The best known of these is the classical extinction paradox, which struggles with the observation that the extinction cross section of a large sphere is twice its geometrical cross sections. There is another apparent paradox associated with extinction of collimated beams. To understand the paradoxes, we must start from first principles and ask where our intuition went wrong. For example, why do we assume that the extinction cross section of a large sphere should be equal to its geometrical cross section? In what follows, we provide simple qualitative arguments that explain the paradoxes and review the relevant literature.

6.4.1 Classical extinction paradox

After Mie has developed in 1908 the mathematical theory of scattering of plane electromagnetic waves from homogeneous spheres, calculations that were not feasible before became, in principle, possible. However, the Mie theory is analytically complicated and, before the age of computers, carrying out the calculations to high orders was a tedious work in which errors were common. Therefore, significant time has passed before Mie theory was applied to electrically large spheres. The result turned out to be unexpected. In 1943, LaMer published a report [25] in which it was suggested that the Mie theory is wrong, at least, as applied to natural unpolarized light, as it predicts a result that is twice as large as the experimental value. The first consistent explanation of this discrepancy (in favor of Mie) was given by Brillouin in 1949 [77], but the subject continues to attract attention [17,78,79]. In particular, in 2011, Berg *et al.* suggested that Brillouin's explanation is incomplete and in some

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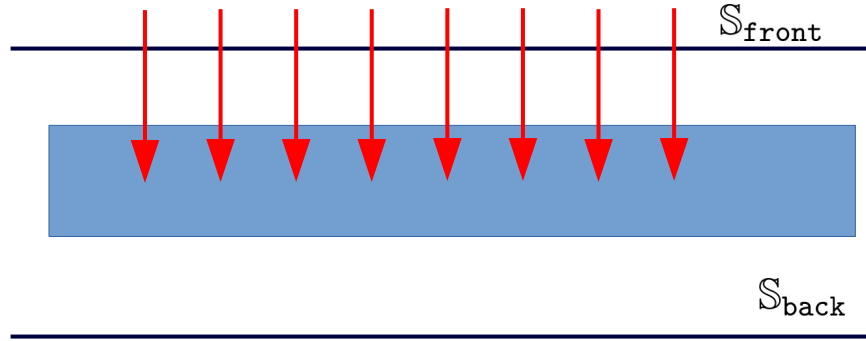[25] Cited by Brillouin in [77] but not readily available at present.

Fig. 9 Illustration of the classical extinction paradox for an opaque slab. The total field at $\mathbb{S}_{back}$ is zero.

instances problematic, and provided a new explanation of the paradox connecting it to the Ewald-Oseen extinction theorem [17].

The question of which explanation is more correct is however moot since there is no reason to believe that the extinction cross section should be equal to the geometrical cross section, and there is nothing strange in the former being twice the latter. In particular, what was referred to as the experimental measurement of the extinction cross section by LaMer was not such; rather, it was a measurement of the geometrical shadow area created by a large sphere. To understand why the extinction cross section can be twice the geometrical cross section, consider the simple physical setup illustrated in Fig. 9. Here we have a wide-front plane wave incident on an opaque slab [26]. Some part of the incident field is reflected back, some is absorbed, but there is virtually no transmission, so that the total field on the measurement surface $\mathbb{S}_{back}$ is zero.

In the physical setup of Fig. 9, we can define extinguished power per unit area. It is easy to show that, in agreement with Sec. 6.2, the energy current $\overline{\mathbf{S}}_{ext}(\mathbf{r})$ is zero on $\mathbb{S}_{front}$. To see this, it suffices to take an arbitrary model for the slab (as long as it is homogeneous in the lateral direction) and compute the reflected field according to the Fresnel formulas. Consequently, the specific (per unit area) extinguished power must be computed on $\mathbb{S}_{back}$ where the total field is zero. However, neither the incident nor the scattered fields are zero on $\mathbb{S}_{back}$. Rather, these two fields are opposite and cancel each other to produce the zero total field in agreement with the Ewald-Oseen extinction theorem. We have therefore:

$$\mathbf{E}_{sca}(\mathbf{r},t) = -\mathbf{E}_{inc}(\mathbf{r},t) \, , \quad \mathbf{B}_{sca}(\mathbf{r},t) = -\mathbf{B}_{inc}(\mathbf{r},t) \qquad \text{for } \mathbf{r} \in \mathbb{S}_{back} \, . \qquad (6.16)$$

Using this in (3.12), we find that

---

[26] Of course, there are no infinite slabs in nature, but, if the slab is sufficiently wide, we can neglect the edge effects.

$$\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) = -\frac{c}{4\pi} \times 2 \times \langle \mathbf{E}_{\text{inc}}(\mathbf{r}, t) \times \mathbf{B}_{\text{inc}}(\mathbf{r}, t) \rangle = -2\,\overline{\mathbf{S}}_{\text{inc}}\,, \quad \mathbf{r} \in \mathbb{S}_{\text{back}}\,. \quad (6.17)$$

Consequently, for the dimensionless specific cross section of extinction $d\sigma_{\text{sca}}/d\mathcal{A}$ where $d\mathcal{A}$ is an element of area, we find

$$\frac{d\sigma_{\text{ext}}}{d\mathcal{A}} = \frac{-\overline{\mathbf{S}}_{\text{ext}}(\mathbf{r}) \cdot \mathbf{e}_z\Big|_{\mathbf{r} \in \mathbb{S}_{\text{back}}}}{\overline{\mathbf{S}}_{\text{inc}} \cdot \mathbf{e}_z} = 2\,. \quad (6.18)$$

This equation indicates that, for every geometrical cross section $\mathcal{A}$ of the slab, the incident energy flux (units of power) is $(\overline{\mathbf{S}}_{\text{inc}} \cdot \mathbf{e}_z)\mathcal{A}$ but the extinguished energy flux *entering the slab* is twice that amount. We thus see that the classical extinction paradox is indeed closely related to the Ewald-Oseen extinction theorem, just as suggested in Ref. [17]. What may seem to be paradoxical in this result is that it appears to contradict the intuitive idea of measuring extinction with a flat detector placed in $\mathbb{S}_{\text{back}}$. Indeed, if we perform such a measurement with and without the slab, and then take the difference, we will measure the incident energy flux times the area of the detector, not twice that amount. But, as discussed in Sec. 6.3, in order for such a measurement to really yield extinction, the detector must be placed deep in the Fraunhofer diffraction zone of the target. Since the target in Fig. 9 is infinite, the Fraunhofer diffraction zone is infinitely far, and a measurement of extinction with a flat detector is impossible (or poorly defined) in this setup. Therefore, there is really no contradiction and no paradox.

To understand the paradox for a finite target, it is instructive to consider a macroscopic mirror. The relevant experiment is illustrated in Fig. 10. Here a wide-front plane wave is incident on a triangular mirror, which reflects the radiation at a slight angle to the backward direction. The figure depicts three field components: the incident field, which, according to the superposition principle, is independent of the target, the scattered field and the total field, which is the sum of the former two components. Importantly, there is a shadow behind the mirror where the incident and scattered field cancel each other almost precisely, similarly to the case of a slab. The small angle that the reflected field makes with the backward direction is unimportant: it was introduced for a more convenient graphical representation of the fields. Now, it is clear that the power that is reflected back by the mirror is approximately equal to the incident field power intercepted by the mirror surface. In addition to that, the scattered field includes the forward component (the Ewald-Oseen field), which cancels the incident field in the geometrical shadow. Therefore, the scattering cross section of the mirror is twice its geometrical cross section. For a perfect mirror, scattering is equal to extinction. To obtain the same result as a measurement with a flat power-integrating detector, we would need to place the latter deep in the Fraunhofer diffraction zone of the mirror where the geometrical shadow is no longer present. To appreciate the involved scales, consider the following example. For a mirror of the transverse size $1\,\text{cm}$ at the wavelength of $1\,\mu\text{m}$, distance to the detector must be much larger than $100\,\text{m}$, probably of the scale of $1\,\text{km}$ or more. Yet, if we could arrange

(a) Incident Field

(b) Scattered Field

Ewald-Oseen Field
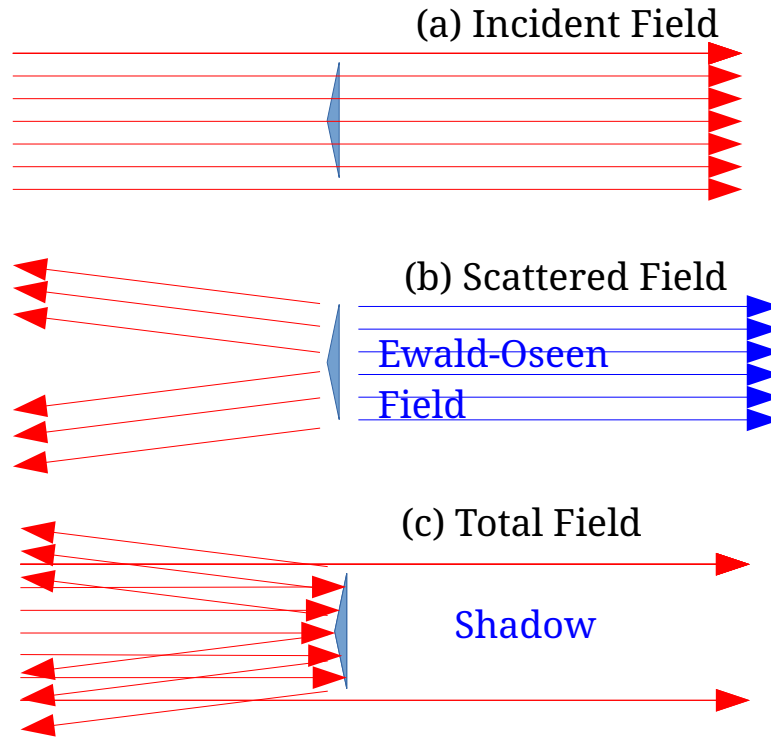
(c) Total Field

Shadow

Fig. 10 Illustration of the classical extinction paradox for a finite target. A triangular mirror reflects incident plane wave at a slight angle to the backward direction. A shadow region is formed behind the mirror. The panels depict schematically the incident (a), the reflected (b) and the total (c) fields. Sketch is limited to the Fresnel diffraction zone in which a geometrical shadow is formed. At larger distances, the sketch (which is, essentially, based on geometrical optics) is not accurate.

such an experiment, the flat detector of appropriate size would measure correctly (in the differential scheme of Sec. 6.3) the extinction cross section of the mirror.

The basic argument explained above goes back to the work of Brillouin [77]. Note that we have not proved that the extinction cross section of all targets that are much larger than the wavelength tends to twice their geometrical cross sections, although this is, in fact, the case. Rather, we have shown that there is nothing surprising in extinction cross section being twice the geometrical cross section, and therefore there is no paradox. If the mirror in the above example is not perfect, or if the target is a large transparent object, which does not reflect back all incident radiation and does not form a well-defined shadow, the extinction cross section still tends to be twice the geometrical cross section, although this is not easy to prove using only arguments of geometrical optics. We refer the reader for further details to Ref. [17] and the more recent review [75].

6.4.2 Paradox of a collimated beam

This paradox is less known but deserves some discussion. Consider again the setup of Fig. 8 where we take the differential measurement with and without the target with Detector 1 while Detector 2 is not present. The paradox is that the same measurement is obtained regardless of position of Detector 1 on the axis of the beam. We can, for example, move the detector twice further from the target and would obtain the same measurement. This seems to be contradictory because the amplitude of the incident field of a well-collimated beam is constant while the amplitude of the scattered field scales as $1/L$ where L is the distance between the target and Detector 1. Since the extinguished power current contains one copy of the scattered field, one might conclude that the measurement should also fall off as $1/L$. However, no matter where we place the detector along the beam, we obtain the same power measurement.

An intuitive but incorrect explanation of this paradox relies on the observation that the beam cannot be perfectly collimated but always diverges. Therefore, the further we move the detector, the larger is the interaction area. Even though the scattered field scales as $1/L$, the total power flux intercepted by the detector surface remains constant due to the increased integration area. To see that this explanation is insufficient, consider a Gaussian beam of waist a, and let the target be located in the waist. The divergence of the beam in the Fresnel diffraction zone $L \leq ka^2$ is very slow and definitely not significant enough to compensate for the $1/L$ scaling of the scattered field. The consideration here is somewhat muddled by the complicated mathematical structure of vector Gaussian beams, but all mathematical models, regardless of the level of approximation, will indicate that divergence of the beam is not enough to explain the paradox.

In Ref. [74], an explanation was given for the case of scalar Gaussian beams whose mathematical modeling is much easier. The paradox however remains the same. It was shown that, although the divergence of the beam may be negligible in the Fresnel diffraction zone, the energy current $\overline{\mathbf{S}}_{\text{ext}}(\rho, z) \cdot \mathbf{e}_z$ is an oscillatory function of ρ, where ρ is the radial coordinate in a cylindrical reference frame whose Z-axis coincides with the optical axis of the beam. The oscillations becomes slower when z is increased so that the integral

$$\int\limits_{\mathbb{S}_{\text{det}}} \overline{\mathbf{S}}_{\text{ext}}(\rho, z) \cdot \mathbf{e}_z \, \rho \mathrm{d}\rho \tag{6.19}$$

remains constant independently of z. This consideration is applicable only in Fresnel diffraction zone. At larger target-detector separations, the beam indeed starts to diverge, and the argument based on increasing integration area becomes also relevant.

While it is difficult and, perhaps, not so interesting to show in detail how this oscillatory behavior works for various vectorial electromagnetic beams, the simple model of Ref. [74] shows that there is really no paradox. We can be sure from energy conservation that measuring the extinguished power in the setup of Fig. 8 will always yield the correct result as long as the detector surface intercepts almost all power of the incident beam.

7 Some auxiliary formulas and examples

7.1 Lorentz model of constitutive relations in time and frequency domain

Let the target be spatially-uniform so that we can omit $\mathbf{r}$ in the list of arguments and focus of the temporal and frequency dependence of various linear response coefficients. Consider the well-known Lorentz dispersion formula for the susceptibility,

$$\chi(\omega) = \frac{\omega_0^2 \, \chi_0}{\omega_0^2 - \omega^2 - \mathrm{i}\,\gamma\omega} \,, \quad \chi_0 = \frac{\epsilon_0 - 1}{4\pi} \,, \tag{7.1}$$

where ϵ_0 and χ_0 are the static values of the permittivity and susceptibility at $\omega = 0$, ω_0 is the resonance frequency, γ is the relaxation constant, and, for simplicity, let $\omega_0 > \gamma/2 > 0$. Applying the inversion formula (3.6b) to (7.1), we find the time-domain response function,

$$\chi(\tau) = \Theta(\tau) \frac{\omega_0^2 \, \chi_0 \, e^{-(\gamma/2)\tau}}{\sqrt{\omega_0^2 - (\gamma/2)^2}} \sin\left(\tau\sqrt{\omega_0^2 - (\gamma/2)^2}\right) \,, \tag{7.2}$$

where $\Theta(x)$ is the unit step function. It can be seen that $\chi(\tau)$ is zero for all $\tau < 0$, just as expected. However, equation (7.2) illustrates an additional point. Namely, $\chi(\tau)$ is also continuous at $\tau = 0$ and its first derivative is bounded (but may be discontinuous). Physically, the condition $\chi(\tau = 0) = 0$ implies that the electric field cannot induce a current in the medium instantaneously; there is always some lag time. Continuity of $\chi(\tau)$ can be proved more generally assuming that $\chi(\omega)$ approaches zero faster than $1/|\omega|$ when $|\omega| \to \infty$. This is the case for all practically-relevant constitutive relations.

We next adduce similar expressions for the coupling functions $\kappa(\omega)$ and $\kappa(\tau)$. We have from the definition (4.18)

$$\kappa(\omega) = \frac{\omega_0^2 \, \chi_0}{\Omega_0^2 - \omega^2 - \mathrm{i}\,\gamma\omega} \,. \tag{7.3}$$

where

$$\Omega_0^2 = \left(1 + \frac{4\pi}{3}\chi_0\right)\omega_0^2 = \frac{\epsilon_0 + 2}{3}\omega_0^2 \,. \tag{7.4}$$

Thus, the only difference between $\chi(\omega)$ and $\kappa(\omega)$ (within the Lorentz dispersion model) is that ω_0^2 is replaced by Ω_0^2. Therefore, in time domain we have

$$\kappa(\tau) = \Theta(\tau) \frac{\omega_0^2 \, \chi_0 \, e^{-(\gamma/2)\tau}}{\sqrt{\Omega_0^2 - (\gamma/2)^2}} \sin\left(\tau\sqrt{\Omega_0^2 - (\gamma/2)^2}\right) \,. \tag{7.5}$$

Just like $\chi(\tau)$, $\kappa(\tau)$ is causal and continuous at $\tau = 0$.

We finally compute the time-domain quasi-static polarizability for a target whose material is a Lorentz medium, assuming the dipole moments of the modes $\mathbf{D}_n$ and the eigenvalues $1/\kappa_n$, as defined in (5.109) through (5.113), are known and $\kappa(\omega)$ is given by (7.3). We then have from (5.116):

$$\alpha(\omega) = \sum_n \mathbf{D}_n \otimes \mathbf{D}_n \frac{\omega_0^2 \, \chi_0}{\omega_n^2 - \omega^2 - i\, \gamma \omega} \ . \tag{7.6}$$

where

$$\omega_n^2 = \left[1 + (4\pi/3 - 1/\kappa_n) \, \chi_0\right] \omega_0^2 \ . \tag{7.7}$$

Equivalently, we can introduce the generalized depolarization coefficients ν_n according to

$$\nu_n = 1/3 - 1/4\pi\kappa_n \tag{7.8}$$

so that the resonance frequencies ω_n are given by

$$\omega_n^2 = \left[1 + 4\pi\nu_n\chi_0\right] \omega_0^2 = \left[1 - \nu_n + \nu_n\epsilon_0\right] \omega_0^2 \ . \tag{7.9}$$

For ellipsoids, there are only three non-zero terms in (7.6) and the corresponding depolarization coefficients satisfy $\nu_1 + \nu_2 + \nu_3 = 1$ [80, § 8]. In particular, for spheres, $\nu_1 = \nu_2 = \nu_3 = 1/3$. For $\mathbb{V}$ of a general shape, the number of non-zero terms in (7.6) is infinite and the sum rule $\sum_n \nu_n = 1$ does not hold. Although we present no mathematical proof here, an argument can be given (essentially, based on stability of solutions) that the generalized depolarization coefficients satisfy $0 \le \nu_n \le 1$ (just like in the case for ellipsoids) so that $-8\pi/3 < 1/\kappa_n < 4\pi/3$ [68]. From this, we see that all frequencies ω_n in (7.10) are real.

Computing the Fourier transform of (7.6), we find the time-domain polarizability:

$$\alpha(\tau) = \Theta(\tau) \sum_n \mathbf{D}_n \otimes \mathbf{D}_n \frac{\omega_0^2 \, \chi_0 \, e^{-(\gamma/2)\tau}}{\sqrt{\omega_n^2 - (\gamma/2)^2}} \sin\left(\tau\sqrt{\omega_n^2 - (\gamma/2)^2}\right) \ . \tag{7.10}$$

This expression is of the form (5.117) with τ_n given by

$$\tau_n = \frac{\sqrt{\omega_n^2 - (\gamma/2)^2}}{\omega_0^2 \, \chi_0} \ . \tag{7.11}$$

7.2 Comparison of the conventional definition of the coherence matrix to the one adopted in this chapter

Conventionally, the coherence matrix is defined in terms of the complex analytic signal, which, by design, should correspond to the real-valued physical field. Of course, if the analytic signal is known, the physical field can be computed simply by taking the real part. However, determining the analytic signal corresponding to a given physical field is only possible if the latter has a temporal Fourier transform, at least in terms of generalized functions. Fields consisting of discrete monochromatic components satisfy this condition, but more general stationary stochastic fields do not. Therefore, if some second-order correlation function of a physical field is measured experimentally, the result cannot be mathematically related to the conventional definition of the coherence function.

The above difficulty is rarely mentioned in the literature. However, it was discussed by Wolf in [11] (see Endnote 8). It was suggested that one can start with the real-valued physical fields, use those to compute the real-valued coherence matrix $\Gamma^{(r)}(\mathbf{r}, \mathbf{r}'; \tau)$, and then compute the corresponding complex analytic coherence matrix $\Gamma^{(c)}(\mathbf{r}, \mathbf{r}'; \tau)$. The last step is mathematically sound since $\Gamma^{(r)}(\mathbf{r}, \mathbf{r}'; \tau)$ has a well-defined Fourier transform with respect to the temporal variable τ. Then we have the relation $\Gamma^{(r)}(\mathbf{r}, \mathbf{r}'; \tau) = \text{Re}[\Gamma^{(c)}(\mathbf{r}, \mathbf{r}'; \tau)]$. However, except for some special cases, the described manipulation is not equivalent to using the analytic signal in place of the physical field to define the coherence matrix. Indeed, the analytic signal corresponding to a stochastic stationary field simply does not exist, so an equivalence between the two approaches cannot exist either. Secondly, computing $\Gamma^{(c)}(\mathbf{r}, \mathbf{r}'; \tau)$ requires numerical evaluation of the Hilbert transform of $\Gamma^{(r)}(\mathbf{r}, \mathbf{r}'; \tau)$ with respect to τ. If $\Gamma^{(r)}(\mathbf{r}, \mathbf{r}'; \tau)$ is measured with a finite precision rather than given by a known analytical formula, this step is ill-conditioned and, in any event, is rarely if ever taken in practice. Most importantly, the manipulation is simply unnecessary as it does not convey any new information or increase generality.

To avoid the mathematical complications briefly described above, we define the coherence matrix in terms of the physical real-valued fields. That is, the matrix Γ defined in (5.1) is equivalent to $\Gamma^{(r)}$ of Ref. [11]. We do not introduce of use anywhere the corresponding analytic signal. While there is no loss of generality in this approach, it is useful to note some differences, which mainly concern the degree of coherence γ, which is defined below in (7.12). In the conventional approach, γ is of magnitude 1 for all monochromatic fields. However, if we define Γ in terms of the real-valued fields, the above simple property is lost. Perhaps, this is one of the motivations for using the analytic signal in the definition of Γ.

Another, more trivial departure from the convention is that we define the cross-spectral density $\mathbb{W}$ in (5.4) without the overall factor $1/2\pi$. This is done for consistency with other definitions. For example, the frequency-domain susceptibility $\chi(\mathbf{r}, \omega)$ is usually defined as the Fourier transform of the time-domain influence function $\chi(\mathbf{r}, \tau)$ as in (3.6a), without the $1/2\pi$ factor, which then shows up in the inverse Fourier transform (3.6b).

Before proceeding, we fix a few notations. In addition to the coherence matrix $\Gamma(\mathbf{r}, \mathbf{r}'; \tau)$ (5.1), the cross-spectral density $\mathbb{W}(\mathbf{r}, \mathbf{r}'; \omega)$ (5.4), the intensity $I(\mathbf{r})$ (5.3) and the power spectrum $S(\omega)$ (5.7), we will compute the temporal degree of coherence $\gamma(\mathbf{r}, \mathbf{r}'; \tau)$, which is defined by the equation

$$\gamma^2(\mathbf{r}, \mathbf{r}'; \tau) = \frac{\|\Gamma(\mathbf{r}, \mathbf{r}'; \tau)\|^2}{I(\mathbf{r})I(\mathbf{r}')} , \tag{7.12}$$

where $\| \cdot \|$ denotes the Frobenius matrix norm. The computations will be done for monochromatic fields according to the conventional definitions and the definitions of this chapter. As noted above, any physical, real-valued monochromatic field oscillating at the frequency ω_0 can be written in the form

$$\mathbf{E}(\mathbf{r}, t) = \text{Re}[\boldsymbol{V}(\mathbf{r}, t)] , \tag{7.13a}$$

where

$$\boldsymbol{V}(\mathbf{r}, t) = \boldsymbol{E}(\mathbf{r}) e^{-\mathrm{i}\,\omega_0 t} \tag{7.13b}$$

is the complex analytic signal. We will now compute the quantities listed above according to the two definition for an electric field of the form (7.13).

7.2.1 Conventional Definitions

In the conventional approach, the analytic signal (7.13b) is used in place of the physical electric field in all definitions. Correspondingly, elements of the coherence matrix (5.1) are given by

$$\Gamma_{ij}(\mathbf{r}, \mathbf{r}'; \tau) = \langle V_i^*(\mathbf{r}, t) V_j(\mathbf{r}', t + \tau) \rangle . \tag{7.14}$$

Substituting (7.13b) into (7.14) and performing time averaging (in this case, trivially), we obtain

$$\Gamma_{ij}(\mathbf{r}, \mathbf{r}'; \tau) = E_i^*(\mathbf{r}) E_j(\mathbf{r}') e^{-\mathrm{i}\,\omega_0 \tau} . \tag{7.15}$$

According to (5.3), the field intensity at any point in space is given by

$$I(\mathbf{r}) = \sum_i E_i^*(\mathbf{r}) E_i(\mathbf{r}) = |\boldsymbol{E}(\mathbf{r})|^2 . \tag{7.16}$$

The Frobenius norm of Γ is

$$\|\Gamma(\mathbf{r}, \mathbf{r}'; \tau)\|^2 = \sum_{ij} |E_i^*(\mathbf{r}) E_j(\mathbf{r}') e^{-\mathrm{i}\,\omega_0 \tau}|^2 = |\boldsymbol{E}(\mathbf{r})|^2 \, |\boldsymbol{E}(\mathbf{r}')|^2 . \tag{7.17}$$

Substituting this expression into (7.12), we find that the squared degree of coherence is 1 for all arguments, viz,

$$\gamma^2(\mathbf{r}, \mathbf{r}'; \tau) = 1 \ . \tag{7.18}$$

Finally, the elements of the cross-spectral density matrix (5.4) are

$$W_{ij}(\mathbf{r}, \mathbf{r}'; \omega) = E_i^*(\mathbf{r}) E_j(\mathbf{r}') \delta(\omega - \omega_0) \tag{7.19}$$

and the power spectrum (5.7) is

$$S(\mathbf{r}, \omega) = |\boldsymbol{E}(\mathbf{r})|^2 \delta(\omega - \omega_0) \ . \tag{7.20}$$

It can be seen why the conventional definition is appealing: it predicts that the degree of coherence is of magnitude 1 for any monochromatic field. Conversely, it can be proved that, if the conventional degree of coherence is of magnitude 1 for all of its arguments, then the field is necessarily monochromatic. Also, it seems convenient that the conventional power spectrum is non-zero only for positive frequencies.

7.2.2 Definitions adopted in this chapter

In contrast to the conventional approach, we use the real-valued physical field in all definitions. For the field of the form (7.13), elements of the coherence matrix are computed as

$$\Gamma_{ij}(\mathbf{r}, \mathbf{r}'; \tau) = \langle [\mathbf{E}(\mathbf{r}, t)]_i \ [\mathbf{E}(\mathbf{r}', t + \tau)]_j \rangle \ , \tag{7.21}$$

where $\mathbf{E}(\mathbf{r}, t)$ is the real-valued physical field. Substituting (7.13a) into (7.21) and performing time averaging, we obtain

$$\Gamma_{ij}(\mathbf{r}, \mathbf{r}'; \tau) = \frac{1}{2} \text{Re} \left[E_i^*(\mathbf{r}) E_j(\mathbf{r}') e^{-\mathrm{i}\, \omega_0 \tau} \right] \ . \tag{7.22}$$

Disregarding the insignificant factor of $1/2$, the conventional expression for the coherence matrix (7.15) is, indeed, the analytic signal corresponding to the real coherence matrix defined in (7.22), just as suggested in Ref. [11]. However, this identification can be made only for monochromatic fields. For stochastic non-monochromatic fields, neither $\boldsymbol{V}(\mathbf{r}, t)$ nor $\boldsymbol{E}(\mathbf{r}, t)$ are defined, and (7.22) is not applicable. Instead of this formula, we should use (7.21) to compute the coherence matrix according to the definition of this chapter.

Returning to the monochromatic case, the field intensity (5.3) at any point in space is given by

$$I(\mathbf{r}) = \frac{1}{2} |E(\mathbf{r})|^2 \ . \tag{7.23}$$

The Frobenius norm of Γ is

$$\|\Gamma(\mathbf{r}, \mathbf{r}'; \tau)\|^2 = \sum_{ij} \Gamma_{ij}^2(\mathbf{r}, \mathbf{r}'; \tau)$$

$$= \frac{1}{8}\mathrm{Re}\left\{[\boldsymbol{E}(\mathbf{r})]^2[\boldsymbol{E}^*(\mathbf{r}')]^2 e^{2\mathrm{i}\,\omega_0\tau} + |\boldsymbol{E}(\mathbf{r})|^2|\boldsymbol{E}(\mathbf{r}')|^2\right\} \ . \quad (7.24)$$

In this expression, $\boldsymbol{E}^2 = \boldsymbol{E}\cdot\boldsymbol{E} = \sum_i E_i^2$ and $|\boldsymbol{E}|^2 = \boldsymbol{E}^*\cdot\boldsymbol{E} = \sum_i |E_i|^2$. It can be seen that the Frobenius norm (7.24) is more complicated than the corresponding conventional expression (7.17).

The degree of coherence is computed by combining (7.24) and (7.23) in the definition (7.12), which results in

$$\gamma^2(\mathbf{r}, \mathbf{r}'; \tau) = \frac{1}{2}\left[1 + \mathrm{Re}\frac{[\boldsymbol{E}(\mathbf{r})]^2\,[\boldsymbol{E}^*(\mathbf{r}')]^2}{|\boldsymbol{E}(\mathbf{r})|^2\,|\boldsymbol{E}(\mathbf{r}')|^2}e^{2\mathrm{i}\,\omega_0\tau}\right] \ . \quad (7.25)$$

The magnitude of this expression is not always equal to 1. For example, in the case of a linearly-polarized plane wave propagating along the Z-axis, we have $\gamma^2(z_1, z_2; \tau) = \sin^2[(\omega_0/c)(z_2 - z_1) - \omega_0\tau]$. In this case, γ^2 fluctuates between 0 and 1. For a circularly-polarized plane wave, $\gamma^2(z_1, z_2; \tau) = 1/2$. These results may seem to contradict the intuitive understanding of perfect coherence, but we should keep in mind that γ is not a directly-measurable quantity and expressions for measurable quantities in terms of γ such as the visibility function in an interference experiment depend on the adopted definition of the coherence matrix.

Finally, elements of the cross-spectral density and the power spectrum are

$$W_{ij}(\mathbf{r}, \mathbf{r}'; \omega) = \frac{\pi}{2}\left[E_i^*(\mathbf{r})E_j(\mathbf{r}')\delta(\omega - \omega_0) + E_i(\mathbf{r})E_j^*(\mathbf{r}')\delta(\omega + \omega_0)\right] \quad (7.26)$$

and

$$S(\mathbf{r}, \omega) = \frac{\pi}{2}|\boldsymbol{E}(\mathbf{r})|^2\left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right] \ . \quad (7.27)$$

Even though the power spectrum (7.27) is non-zero for both positive and negative frequencies, expressions for physical observables given in this chapter involve integration over positive frequencies only. Note that we can write

$$W_{ij}(\mathbf{r}, \mathbf{r}'; \omega) = \frac{\pi}{2}\left[w_{ij}(\mathbf{r}, \mathbf{r}'; \omega_0)\,\delta(\omega - \omega_0) + w_{ij}(\mathbf{r}, \mathbf{r}'; -\omega_0)\,\delta(\omega + \omega_0)\right] \ ,$$
$$(7.28)$$

where

$$w_{ij}(\mathbf{r}, \mathbf{r}'; \omega_0) = E_i^*(\mathbf{r})E_j(\mathbf{r}') \ , \quad w_{ij}(\mathbf{r}, \mathbf{r}'; -\omega_0) = E_i(\mathbf{r})E_j^*(\mathbf{r}') \ . \quad (7.29)$$

In the case when $\boldsymbol{E}(\mathbf{r})$ is a plane wave of the form (4.27), we have

$$w_{ij}(\mathbf{r}, \mathbf{r}'; \omega_0) = A_i^* A_j e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}'-\mathbf{r})} \ , \quad w_{ij}(\mathbf{r}, \mathbf{r}'; -\omega_0) = A_i A_j^* e^{\mathrm{i}\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')} \ . \quad (7.30)$$

The function $w_{ij}(\mathbf{r}, \mathbf{r}'; \omega_0)$ (denoted without the label ω_0) appears in the derivations of Sec. 4.6.

7.3 Proof that $\mathbb{W}(\omega)$ is non-negative definite

To prove that $\mathbb{W}(\omega)$ is non-negative-definite, we must show that, for an arbitrary complex vector field $\boldsymbol{F}(\mathbf{r}) \in \mathcal{H}[\mathbb{V}]$, it holds that

$$\langle \mathrm{F}|\mathbb{W}(\omega)|\mathrm{F}\rangle = \int_{\mathbb{V}} \mathrm{d}^3 r_1 \int_{\mathbb{V}} \mathrm{d}^3 r_2 \ \boldsymbol{F}^*(\mathbf{r}_1) \cdot \mathbb{W}(\mathbf{r}_1, \mathbf{r}_2; \omega)\boldsymbol{F}(\mathbf{r}_2) \geq 0 \ . \qquad (7.31)$$

Utilizing the definition (5.4) of $\mathbb{W}$ in terms of Γ, we can re-write the left-hand side in (7.31) as

$$\langle \mathrm{F}|\mathbb{W}(\omega)|\mathrm{F}\rangle = \int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau \int_{\mathbb{V}} \mathrm{d}^3 r_1 \int_{\mathbb{V}} \mathrm{d}^3 r_2 \ \boldsymbol{F}^*(\mathbf{r}_1) \cdot \Gamma(\mathbf{r}_1, \mathbf{r}_2; \tau)\boldsymbol{F}(\mathbf{r}_2) \ . \qquad (7.32)$$

We further use the definition (5.1) of Γ in terms of the electric field $\mathbf{E}$ to obtain

$$\langle \mathrm{F}|\mathbb{W}(\omega)|\mathrm{F}\rangle = \int_{-\infty}^{\infty} e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau \int_{\mathbb{V}} \mathrm{d}^3 r_1 \int_{\mathbb{V}} \mathrm{d}^3 r_2$$
$$\times \langle [\boldsymbol{F}^*(\mathbf{r}_1) \cdot \mathbf{E}(\mathbf{r}_1, t)] \, [\boldsymbol{F}(\mathbf{r}_2) \cdot \mathbf{E}(\mathbf{r}_2, t + \tau)] \rangle \ . \qquad (7.33)$$

Let

$$f(t) = \int_{\mathbb{V}} \boldsymbol{F}(\mathbf{r}) \cdot \mathbf{E}(\mathbf{r}, t) \, \mathrm{d}^3 r \ . \qquad (7.34)$$

Evidently, $f(t)$ is a complex, scalar, stationary stochastic process. It then follows from (7.33) that

$$\langle \mathrm{F}|\mathbb{W}(\omega)|\mathrm{F}\rangle = w(\omega) = \int_{-\infty}^{\infty} \gamma(\tau)e^{\mathrm{i}\,\omega\tau}\mathrm{d}\tau \ , \qquad (7.35)$$

where

$$\gamma(\tau) = \langle f^*(t)f(t + \tau)\rangle \ . \qquad (7.36)$$

We thus see that $\langle \mathrm{F}|\mathbb{W}(\omega)|\mathrm{F}\rangle$ is the power spectrum $w(\omega)$ of a scalar stochastic process $f(t)$. It is well known that the power spectrum cannot be negative in any finite interval of frequencies. This can be shown as follows.

Assume that $w(\omega)$ is negative for $\omega \in [a, b]$ where $b > a$. We can apply a linear filter $\ell(\tau)$ to $f(t)$ so that

$$f_\ell(t) = \int\limits_{-\infty}^{\infty} \ell(\tau) f(t - \tau) \mathrm{d}\tau \ . \tag{7.37}$$

The power spectrum of $f_\ell(t)$ is $w_\ell(\omega) = |\ell(\omega)|^2 w(\omega)$. If $\ell(\omega)$ is an ideal step function equal to 1 in $[a, b]$ and 0 elsewhere (such a filter exists and can be easily defined), then we conclude that

$$\int\limits_{-\infty}^{\infty} w_\ell(\omega)\mathrm{d}\omega = \int\limits_{a}^{b} w(\omega)\mathrm{d}\omega < 0 \ . \tag{7.38}$$

This is an impossibility because

$$\int\limits_{-\infty}^{\infty} w_\ell(\omega)\mathrm{d}\omega = 2\pi\gamma_\ell(0) \tag{7.39}$$

and $\gamma_\ell(0) = \langle f_\ell^*(t) f_\ell(t) \rangle$ is non-negative by definition. If we add the assumption that $w(\omega)$ is continuous, then it cannot be negative at any point, not just in a finite interval.

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