Nonlinearity of Inverse Scattering Problem and the Resolution Limit: Does Multiple Scattering Provide Subwavelength Information?

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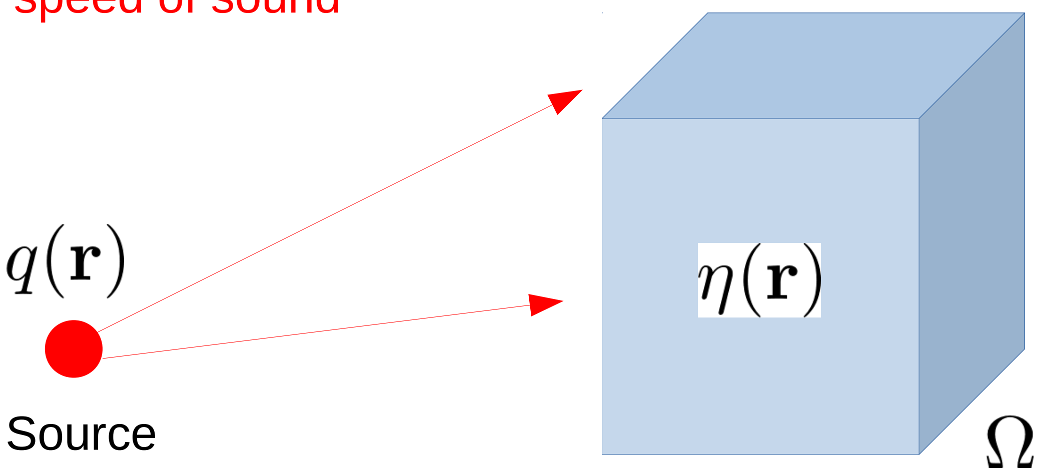
Plan of the talk

1. Inverse scattering problem, Ewald sphere, discretization
2. Toy Problem 1 (3 dof)
3. Toy Problem 2 (4 dof)
4. Toy Problem 3 (N dof)
5. Distorted Born approximation and tangent spaces
6. Model with a realistic interaction and N degrees of freedom (N=2501)

1a. Inverse Scattering Problem (Scalar Waves)

This can be relative speed of sound

$$\eta(\mathbf{r}) = 1 \quad \text{for} \quad \mathbf{r} \notin \Omega$$



We work in frequency domain:

$$U(\mathbf{r}, t) = \mathrm{Re}[u(\mathbf{r}) e^{-i\omega t}]$$

$$k = \frac{\omega}{c} = \mathrm{const}$$

$$\left[\nabla^2 + k^2 \eta(\mathbf{r})\right] u(\mathbf{r}) = -4\pi k^2 q(\mathbf{r})$$

Everywhere in space (**and + boundary conditions at infinity**)

$$\left(\nabla^2 + k^2\right) u(\mathbf{r}) = -4\pi k^2 \left[V(\mathbf{r}) u(\mathbf{r}) + q(\mathbf{r})\right]$$

Identical transformation of the previous equation

$$V(\mathbf{r}) \equiv \frac{\eta(\mathbf{r}) - 1}{4\pi}$$

The "potential" is zero outside of the domain

1b. Inverse Scattering Problem (cont.)

$$\left(\nabla^2 + k^2\right) u(\mathbf{r}) = -4\pi k^2 \left[V(\mathbf{r})u(\mathbf{r}) + q(\mathbf{r})\right]$$

From previous page

$$u(\mathbf{r}) = u_{\text{inc}}(\mathbf{r}) + \underbrace{\int_\Omega G(\mathbf{r}, \mathbf{r}')V(\mathbf{r}')u(\mathbf{r}')d^3r'}_{\text{Scattered field}}$$

Lippmann-Schwinger equation

Where

$$u_{\text{inc}}(\mathbf{r}) = \int G(\mathbf{r}, \mathbf{r}')q(\mathbf{r}')d^3r'$$

This is the incident field; it would be the total field for the given source in the absence of the scatterer.

$$G(\mathbf{r}, \mathbf{r}') = k^2 \frac{\exp\left(ik|\mathbf{r} - \mathbf{r}'|\right)}{|\mathbf{r} - \mathbf{r}'|}$$

The free-space Green's function; it satisfies the radiation (Sommerfeld) boundary conditions at infinity.

1c. Inverse Scattering Problem (T-matrix)

Introduce the "polarization" field: $P(\mathbf{r}) = V(\mathbf{r})u(\mathbf{r})$ (vanishes outside of the domain)

$$u(\mathbf{r}) = u_{\text{inc}}(\mathbf{r}) + \int_{\Omega} G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')u(\mathbf{r}')d^3r'$$

Multiply by V

$$P(\mathbf{r}) = V(\mathbf{r})u_{\text{inc}}(\mathbf{r}) + \int_{\Omega} V(\mathbf{r})G(\mathbf{r},\mathbf{r}')P(\mathbf{r}')d^3r' \quad (\mathbf{r} \in \Omega)$$

From linearity

The T-matrix

$$P(\mathbf{r}) = \int_{\Omega} T(\mathbf{r},\mathbf{r}')u_{\text{inc}}(\mathbf{r}')d^3r'$$

Higher-order terms contain integrals

$$T(\mathbf{r},\mathbf{r}') = V(\mathbf{r})\delta(\mathbf{r},\mathbf{r}') + V(\mathbf{r})G(\mathbf{r},\mathbf{r}')V(\mathbf{r}') + \ldots$$

1d. Inverse Scattering Problem (cont.)

$$P(\mathbf{r}) = \int_\Omega T(\mathbf{r}, \mathbf{r}') u_{\rm inc}(\mathbf{r}') d^3 r'$$

$$T = (I - VG)^{-1} V \xrightarrow{\text{(if converges)}} V + VGV + VGVGV + \ldots$$

$$\begin{aligned} u_{\rm scatt}(\mathbf{r}) &= \int_\Omega G(\mathbf{r}, \mathbf{r}') P(\mathbf{r}') d^3 r' \\ &= \int_\Omega G(\mathbf{r}, \mathbf{r}_1) T(\mathbf{r}_1, \mathbf{r}_2) u_{\rm inc}(\mathbf{r}_2) d^3 r_1 d^3 r_2 \end{aligned}$$

We will next assume that the incident field is a plane wave and make the far-field approximation for the Green's function

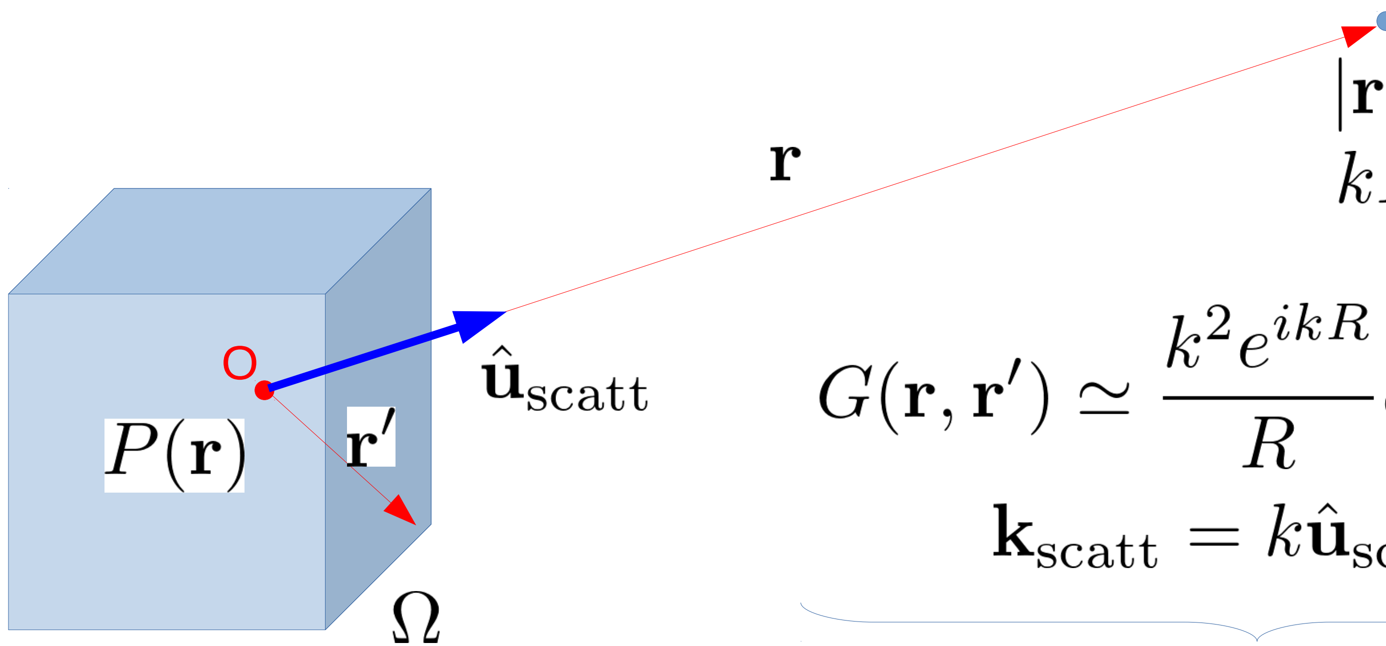
This will lead to loss of high-frequency information about the target

1e. Inverse Scattering Problem (far-field)

$$u_{\text{scatt}}(\mathbf{r}) = \int_\Omega G(\mathbf{r}, \mathbf{r}')P(\mathbf{r}')d^3r'$$

Scattered field (from previous page)

Point of observation in the far field of the scatterer:



$$|\mathbf{r}| = R = \text{const}$$

$$kR \gg 1$$

$$G(\mathbf{r}, \mathbf{r}') \simeq \frac{k^2 e^{ikR}}{R} e^{-i\mathbf{k}_{\text{scatt}} \cdot \mathbf{r}'}$$

$$\mathbf{k}_{\text{scatt}} = k\hat{\mathbf{u}}_{\text{scatt}}$$

Far-field approximation

$$u_{\text{scatt}}(\mathbf{r}) = \frac{k^2 e^{ikR}}{R} \int_\Omega e^{-i\mathbf{k}_{\text{scatt}} \cdot \mathbf{r}} P(\mathbf{r}) d^3r$$

Not a regular Fourier transform of the polarization field; can not be inverted.....

1f. Inverse Scattering Problem (scattering amplitude)

recall that $$P(\mathbf{r}) = \int_\Omega T(\mathbf{r}, \mathbf{r}') u_{\text{inc}}(\mathbf{r}') d^3 r'$$

... and let $u_{\text{inc}}(\mathbf{r}) = e^{i\mathbf{k}_{\text{inc}} \cdot \mathbf{r}}$

$$u_{\text{scatt}}(\mathbf{r}) = \frac{k^2 e^{ikR}}{R} \int_\Omega e^{-i\mathbf{k}_{\text{scatt}} \cdot \mathbf{r}} P(\mathbf{r}) d^3 r$$

$$= \frac{k^2 e^{ikR}}{R} \int_\Omega e^{-i\mathbf{k}_{\text{scatt}} \cdot \mathbf{r}_1} \, T(\mathbf{r}_1, \mathbf{r}_2) \, e^{i\mathbf{k}_{\text{inc}} \cdot \mathbf{r}_2} d^3 r_1 d^3 r_2$$

$$f(\mathbf{k}_{\text{scatt}}, \mathbf{k}_{\text{inc}}) = \int_\Omega T(\mathbf{r}_1, \mathbf{r}_2) \, e^{i(\mathbf{k}_{\text{inc}} \cdot \mathbf{r}_2 - \mathbf{k}_{\text{scatt}} \cdot \mathbf{r}_1)} d^3 r_1 d^3 r_2$$

Scattering amplitude (measurable quantity)

T-matrix (uniquely defined by V)

ISP: Given measurements of scattering amplitude, reconstruct V

1g. Inverse Scattering Problem (linear regime)

$$T(\mathbf{r}_1, \mathbf{r}_2) \approx V(\mathbf{r}_1)\delta(\mathbf{r}_1 - \mathbf{r}_2)$$

This is approximately true in the weak scattering regime when V is in some sense small

$$f(\mathbf{k}_{\text{scatt}}, \mathbf{k}_{\text{inc}}) = \int_\Omega V(\mathbf{r}) e^{i(\mathbf{k}_{\text{inc}} - \mathbf{k}_{\text{scatt}})\cdot\mathbf{r}} d^3r$$

$$|\mathbf{k}_{\text{inc}} - \mathbf{k}_{\text{scatt}}| \leq 2k = \frac{2\omega}{c}$$

Ewald sphere radius

Resolution limit in 3D? Sphere is not a cube!

$$\Delta > \lambda/4 \ , \quad \lambda = 2\pi/k$$

(because a circumscribed cube with side 4k contains the Ewald sphere but has some empty corners (we do not know Fourier data in these regions)

$$\Delta < \sqrt{3}\lambda/4 \ , \quad \lambda = 2\pi/k$$

(because we know Fourier data every where in a cube inscribed inside the Ewald sphere plus some additional data outside of the cube.

1h. Inverse Scattering Problem (nonlinear regime)

$$f(\mathbf{k}_{\mathrm{inc}}, \mathbf{k}_{\mathrm{scatt}}) = \int e^{i\mathbf{k}_{\mathrm{inc}}\cdot\mathbf{r}} V(\mathbf{r}) e^{-i\mathbf{k}_{\mathrm{scatt}}\cdot\mathbf{r}} d^3r$$

$$+ \int e^{-i\mathbf{k}_{\mathrm{scatt}}\cdot\mathbf{r}_1} V(\mathbf{r}_1) G(\mathbf{r}_1, \mathbf{r}_2) V(\mathbf{r}_2) e^{i\mathbf{k}_{\mathrm{inc}}\cdot\mathbf{r}_2} d^3r_1 d^3r_2 + \ldots$$

We can write this expansion in terms of the Fourier transform of the Potential:

$$G(\mathbf{r}, \mathbf{r}') = \int g(\mathbf{q}) e^{i\mathbf{q}\cdot(\mathbf{r}-\mathbf{r}')} d^3q$$

$$f(\mathbf{k}_{\mathrm{inc}}, \mathbf{k}_{\mathrm{scatt}}) = \tilde{V}(\mathbf{k}_{\mathrm{inc}} - \mathbf{k}_{\mathrm{scatt}})$$

$$+ \int \tilde{V}(\mathbf{k}_{\mathrm{inc}} - \mathbf{q}) g(\mathbf{q}) \tilde{V}(\mathbf{q} - \mathbf{k}_{\mathrm{scatt}}) d^3q + \ldots$$

Here we already have Fourier wave vectors outside of the Ewald sphere

1i. Some History

The idea to use nonlinearity of ISP to achieve super-resolution, although in a somewhat implicit form (Chew and coo-authors)

* M. Moghaddam, W. C. Chew, and M. Oristaglio, Int. J. Imaging Syst. Technol. 3, 318, 1991
* M. Moghaddam, W. C. Chew, IEEE Trans. Geosci. Remote Sensing 30, 147, 1992.
* F.-C. Chen and W. C. Chew, Appl. Phys. Lett. 72, 3080, 1998
* T. J. Cui, W. C. Chew, X. X. Yin, and W. Hong, IEEE Trans. Ant. Propag. 52, 1398, 2004.

More explicit claims:

* F. Simonetti, Phys. Rev. E 73, 036619, 2006
* K. Belkebir, P. C. Chaumet, and A. Sentenac, J. Opt. Soc. Am. A 23, 586, 2006
* G. Maire et al., Phys. Rev. Lett. 102, 213905, 2009
* C. Gilmore et al., IEEE Antennas Wireless Propagation Lett. 9, 393, 2010
* T. Zhang et al., Optica 3, 609, 2016 ← Experimental demonstration of resolution $\lambda/10$ (but with strong a priori constraints)

A review article in which the super-resolution in nonlinear ISP is presented as a fact:

M. T. Testorf and M. A. Fiddy, "Superresolution imaging – Revisited," Adv. Imaging Electron Phys. 163, 165, 2010

1j. Discretization

$$\int_\Omega \underbrace{e^{-i\mathbf{k}_{\text{scatt}}\cdot\mathbf{r}_1}}_{\mathtt{A}} \underbrace{T(\mathbf{r}_1,\mathbf{r}_2)}_{\mathtt{T}} \underbrace{e^{i\mathbf{k}_{\text{inc}}\cdot\mathbf{r}_2}}_{\mathtt{B}} d^3r_1 d^3r_2 = \underbrace{f(\mathbf{k}_{\text{scatt}},\mathbf{k}_{\text{inc}})}_{\Phi}$$

Data matrix

$$\mathtt{AT[V]B} = \Phi$$

This is the algebraic statement of the nonlinear ISP. Namely, given

-- the data matrix Φ

-- the measurement matrices A,B

-- the algebraic relation between T and V

find all elements of the diagonal matrix V

$$\mathtt{T[V]} = (\mathtt{I} - \mathtt{VG})^{-1}\mathtt{V}$$

Number of incidence directions

Number of voxels

Number of detection directions

$$N_s \times N \text{ matrix: } \mathtt{A}_{ln} = e^{i\mathbf{k}_l^{(\text{inc})}\cdot\mathbf{r}_n}$$

$$N \times N_d \text{ matrix: } \mathtt{B}_{nm} = e^{-i\mathbf{k}_m^{(\text{scatt})}\cdot\mathbf{r}_n}$$

$$N_s \times N_d \text{ matrix: } \Phi_{lm} = f(\mathbf{k}_l^{(\text{scatt})},\mathbf{k}_m^{(\text{inc})})$$

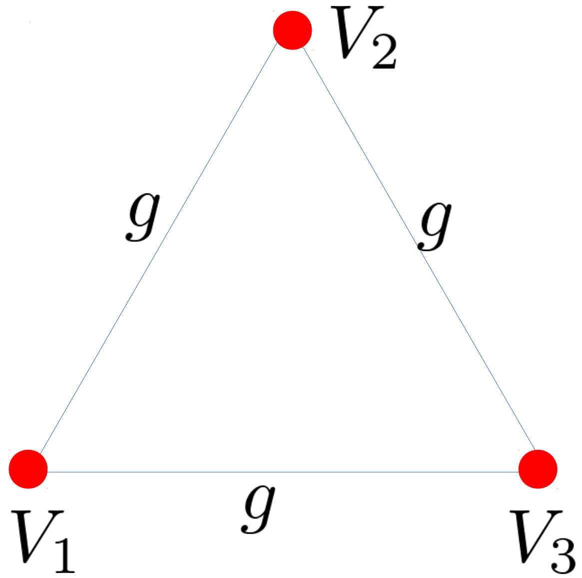
$$N \times N \text{ matrices: } \mathtt{T},\mathtt{G},\mathtt{V} \ (\mathtt{V} \text{ is diagonal})$$

2a. Toy Problem 1 with 3 degrees of freedom

$$\mathtt{G} = \begin{bmatrix} 0 & g & g \\ g & 0 & g \\ g & g & 0 \end{bmatrix}$$

$$T_{nm} = V_n \delta_{nm} + g \frac{\kappa_n \kappa_m}{1 - gS}$$

$$\kappa_n = \frac{V_n}{1 + gV_n} \ , \quad S = \sum_{n=1}^{3} \kappa_n \ .$$



We will construct the measurement matrices A and B from the following three orthogonal basis vectors:

$$\mathtt{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \ , \quad \mathtt{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \ , \quad \mathtt{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

2b. Toy Problem 1

(a) Band-limited measurement (u1 and u2)

$$\mathtt{A} = \begin{bmatrix} \mathtt{u}_1^T \\ \mathtt{u}_2^T \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \end{bmatrix}, \quad \mathtt{B} = \begin{bmatrix} \mathtt{u}_1 & \mathtt{u}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \\ 1 & 1 \end{bmatrix}.$$

(a.i) Linear regime $g=0$

$$\begin{bmatrix} V_1 + V_2 + V_3 & V_1 - 2V_2 + V_3 \\ V_1 - 2V_2 + V_3 & V_1 + 4V_2 + V_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\phi_{12} = \phi_{21} \quad \text{AND} \quad \phi_{22} = 2\phi_{11} - \phi_{12}$$

Condition of physical admissibility of data

If satisfied

$$V_1 + V_2 + V_3 = \phi_{11}$$
$$V_1 - 2V_2 + V_3 = \phi_{12}$$

Two independent equations for three unknowns, hence, the IP is “band-limited”

2c. Toy Problem 1

(a.ii) Non-linear regime $g{=}/{=}0$

$$\begin{bmatrix} \frac{\kappa_1+\kappa_2+\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} & \frac{\kappa_1-2\kappa_2+\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} \\ \frac{\kappa_1-2\kappa_2+\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} & \kappa_1+4\kappa_2+\kappa_3+\frac{g(\kappa_1-2\kappa_2+\kappa_3)^2}{1-g(\kappa_1+\kappa_2+\kappa_3)} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\kappa_n = \frac{V_n}{1+gV_n}$$

Consistency requires that

$$\phi_{12} = \phi_{21} \quad \text{AND} \quad \phi_{22} = \frac{2\phi_{11} - \phi_{12} + g\phi_{12}^2}{1+g\phi_{11}}$$

Condition of physical admissibility of data

If satisfied

$$\frac{V_1}{1+gV_1} + \frac{V_3}{1+gV_3} = \frac{1}{3}\frac{2\phi_{11}+\phi_{12}}{1+g\phi_{11}}$$

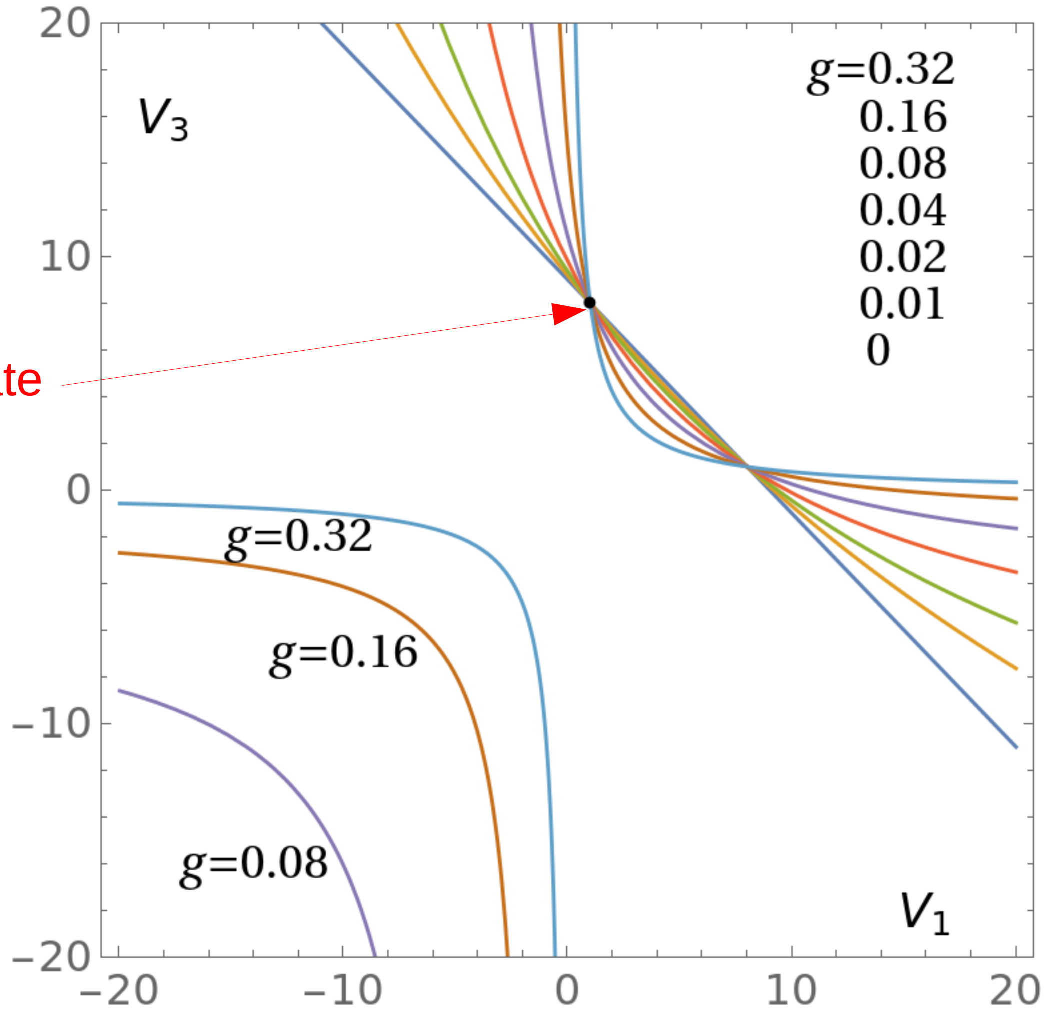
$$V_2 = \frac{\phi_{11}-\phi_{12}}{3+g(2\phi_{11}+\phi_{12})}$$

Still only two independent equations for three unknowns.

Nonlinearity in the ISP did not force uniqueness

2d. Toy Problem 1
Solutions in the case of band-limited measurements

Model value used to generate the data matrix



Loci of all points in the (V1,V3) plane that satisfy the nonlinear equations, assuming the data is physically-admissible (in range of the forward operator) for various values of the interaction parameter *g*.

The data were generated in each case using the same model shown by a circular dot in the plot.

2e. Toy Problem 1

(b) Non-band-limited measurements (u1,u3)

$$\mathtt{A} = \begin{bmatrix} \mathtt{u}_1 \\ \mathtt{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{bmatrix}, \quad \mathtt{B} = \begin{bmatrix} \mathtt{u}_1 & \mathtt{u}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & -1 \end{bmatrix}.$$

(b.i) Linear regime g=0

$$\begin{bmatrix} V_1 + V_2 + V_3 & V_1 - V_3 \\ V_1 - V_3 & V_1 + V_3 \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\phi_{12} = \phi_{21}$$

This is the only physical admissibility condition. Data matrix must be symmetric (reciprocity)

If satisfied

$$V_1^{\mathtt{inv}} = (\phi_{22} + \phi_{12})/2$$

$$V_2^{\mathtt{inv}} = \phi_{11} - \phi_{22}$$

$$V_3^{\mathtt{inv}} = (\phi_{22} - \phi_{12})/2$$

This is the unique inverse solution

2f. Toy Problem 1

(b.ii) Non-linear regime $g \neq 0$

$$\begin{bmatrix} \frac{\kappa_1+\kappa_2+\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} & \frac{\kappa_1-\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} \\ \frac{\kappa_1-\kappa_3}{1-g(\kappa_1+\kappa_2+\kappa_3)} & \kappa_1+\kappa_3+\frac{g(\kappa_1-\kappa_3)^2}{1-g(\kappa_1+\kappa_2+\kappa_3)} \end{bmatrix} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix}$$

$$\phi_{12} = \phi_{21}$$

This condition must still be satisfied (reciprocity)

If satisfied

$$\kappa_1^{\mathtt{inv}} = \frac{1}{2}\left(\phi_{22} + \phi_{12}\frac{1-g\phi_{12}}{1+g\phi_{11}}\right)$$

$$\kappa_2^{\mathtt{inv}} = -\phi_{22} + \frac{\phi_{11}+g\phi_{12}^2}{1+g\phi_{11}}$$

$$\kappa_3^{\mathtt{inv}} = \frac{1}{2}\left(\phi_{22} - \phi_{12}\frac{1+g\phi_{12}}{1+g\phi_{11}}\right)$$

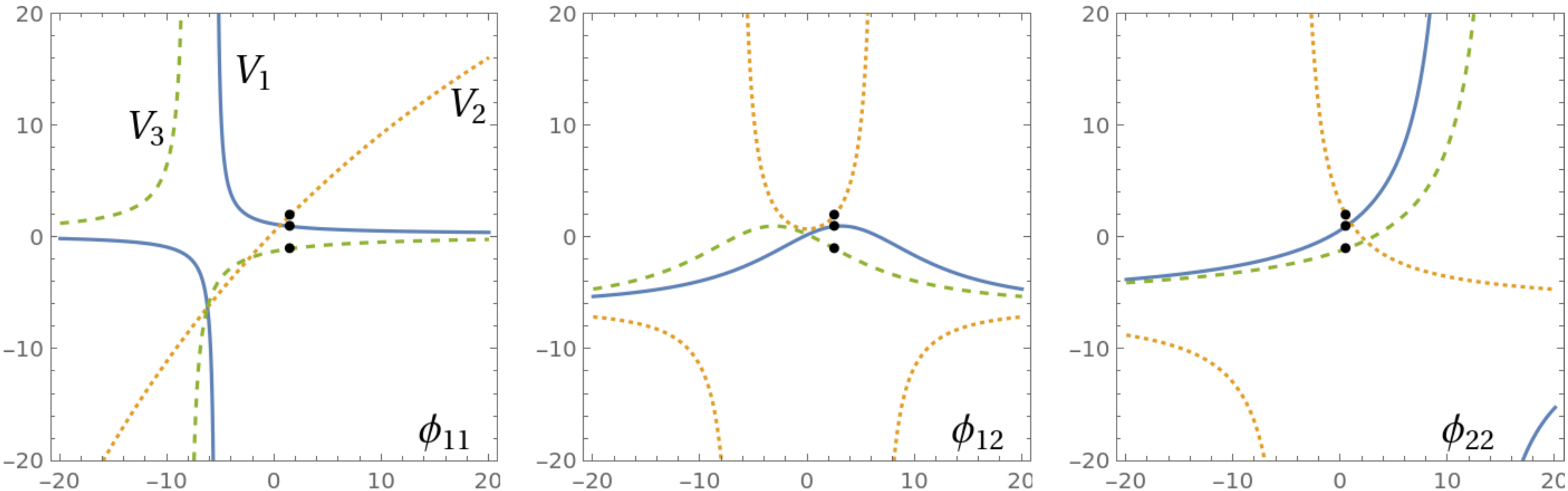
$$V_n^{(\mathtt{inv})} = \kappa_n^{\mathtt{inv}}/(1-g\kappa_n^{\mathtt{inv}})$$

This is the unique nonlinear inverse solution

2g. Toy Problem 1 Solutions in the case of non-band-limited measurements

$$g = 0.1$$

$\phi_{12} \approx 2.5$	$\phi_{11} \approx 1.5$	$\phi_{11} \approx 1.5$
$\phi_{22} \approx 0.5$	$\phi_{22} \approx 0.5$	$\phi_{12} \approx 2.5$



$$(\phi_{11}, \phi_{12}, \phi_{22}) \approx (1.5, 2.5, 0.5) \Leftrightarrow (\mathrm{V}_1, V_2, V_2) = (1, 2, -1)$$

2h. Toy Problem 1 Why the first measurement scheme was band-limited and the second was not? Select a pair u's and form Hadamard products:

u1 and u2 (band-limited scheme):

$$\mathbf{u}_1 \circ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 \circ \mathbf{u}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 \circ \mathbf{u}_2 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$$

Two of these vectors are linearly-independent

u1 and u3 (not a band-limited scheme):

$$\mathbf{u}_1 \circ \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_1 \circ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 \circ \mathbf{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

All three vectors are linearly-independent

In General:

$$\mathrm{rank}[\mathtt{A} * \mathtt{B}] < N$$

→ Linear inverse problem is band-limited (under-determined)

$$\mathrm{rank}[\mathtt{A} * \mathtt{B}] \geq N$$

→ Linear inverse problem is not band-limited (exactly determined or over-determined)

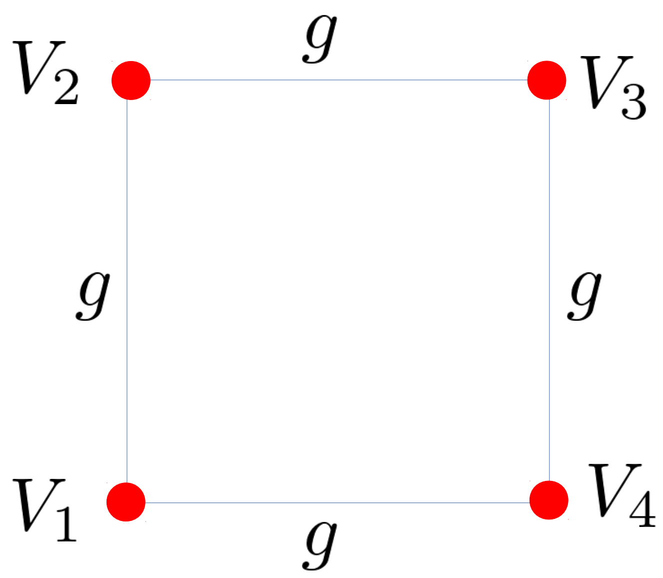
↑ Khatri-Rao product assuming each matrix consists of just one block

$$\mathtt{A}_{in}\mathtt{B}_{nj} = (\mathtt{A} * \mathtt{B})_{(ij),n}$$

3a. Toy Problem 2 with 4 degrees of freedom

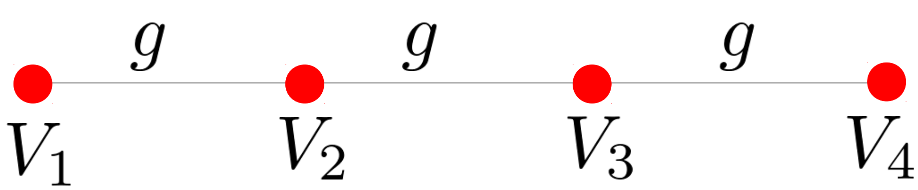
a) Cyclic tight-binding model

$$\mathtt{G} = \begin{bmatrix} 0 & g & 0 & g \\ g & 0 & g & 0 \\ 0 & g & 0 & g \\ g & 0 & g & 0 \end{bmatrix}$$



b) Chain tight-binding model

$$\mathtt{G} = \begin{bmatrix} 0 & g & 0 & 0 \\ g & 0 & g & 0 \\ 0 & g & 0 & g \\ 0 & 0 & g & 0 \end{bmatrix}$$



3b. Toy Problem 2. Measurement matrices

$$\mathtt{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathtt{u}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathtt{u}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathtt{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix}, \quad \mathtt{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

$$\mathtt{rank}[\mathtt{A} * \mathtt{B}] = 3 < N = 4$$

The linear inverse problem is under-determined:
3 linearly-independent equations and 4 unknowns

3c. Toy Problem 2. Linear solution (g=0)

The data matrix is 3x3 and therefore has 9 elements. Physical admissibility conditions are:

$$\phi_{ij} = \phi_{ji}$$

← Reciprocity

$$\phi_{13} = \phi_{23} \;, \quad \phi_{12} + \phi_{22} = 2\phi_{33} \;, \quad \phi_{11} = \phi_{22}$$

← Additional conditions

So, only 3 data matrix elements are independent.

↓ If conditions hold

$$V_1^{\mathtt{inv}} = \frac{1}{2}\left(\phi_{33} + \phi_{23}\right)$$

$$V_3^{\mathtt{inv}} = \frac{1}{2}\left(\phi_{33} - \phi_{23}\right)$$

$$V_2^{\mathtt{inv}} + V_4^{\mathtt{inv}} = \phi_{22} - \phi_{33}$$

So, the linearized solution is indeed non-unique

3d. Toy Problem 2.
Nonlinear solution – Cyclic interaction

$$\frac{\phi_{13}}{\phi_{23}} = \frac{1 + g(\phi_{22} - 3\phi_{33}) + g^2(\phi_{22}\phi_{33} - \phi_{23}^2)}{1 - g\phi_{22} - g^2\phi_{23}^2 - g\phi_{33} + g^2\phi_{22}\phi_{33}}$$

$$\phi_{12} = \frac{2\phi_{33} - \phi_{22} + g(\phi_{23}^2 - \phi_{22}\phi_{33} - \phi_{13}\phi_{23})}{1 - g\phi_{33}}$$

$$\phi_{11}(\phi_{22} - \phi_{33}) = \phi_{12}^2 - \phi_{13}^2 + \phi_{23}(2\phi_{13} - \phi_{23}) + \phi_{33}(\phi_{22} - 2\phi_{12})$$

Admissibility conditions
(in addition to reciprocity)

$\Downarrow$

$$V_1^{\mathtt{inv}} = \frac{1}{2}\frac{(\phi_{12} + \phi_{22})\phi_{33} - (\phi_{13} + \phi_{23})\phi_{23}}{\phi_{12} + \phi_{22} - 2\phi_{33}}$$

$$V_3^{\mathtt{inv}} = \frac{1}{2}\frac{(\phi_{12} + \phi_{22})\phi_{33} - (\phi_{13} + \phi_{23})\phi_{23}}{\phi_{12} + \phi_{22} + 2\phi_{33}}$$

$$V_2^{\mathtt{inv}} + V_4^{\mathtt{inv}} = \frac{\phi_{13}(\phi_{22} - \phi_{23}) + \phi_{23}(\phi_{23} - \phi_{12}) + \phi_{33}(\phi_{12} - \phi_{22})}{\phi_{12} + \phi_{13} + \phi_{22} - 4\phi_{33} + 2g(\phi_{22}\phi_{33} - \phi_{23}^3)}$$

Nonlinear inverse solution
is still non-unique

3e. Toy problem 2.
Nonlinear solution – Tight-binding in a chain

$$V_n^{\mathtt{inv}} = \frac{\mathcal{N}_n}{g\mathcal{D}_n} \ , \quad n = 1, 2, 3, 4$$

$$\mathcal{N}_n = a_n + b_n g + c_n g^2 + d_n g^3 + e_n g^4$$

$$\mathcal{D}_n = p_n + q_n g + r_n g^2 + s_n g^3 + t_n g^4$$

The coefficients are combinations of rational functions and square root and some also depend on g.

Formulas are quite lengthy...

$$a_1 = 2(1-R)\ ;$$
$$b_1 = 2\big[\phi_{11} + 8\phi_{23} + 7\phi_{33} - 2\phi_{22} + (3\phi_{33} - \phi_{23})R\big]\ ;$$
$$c_1 = \phi_{11}(6\phi_{23} - \phi_{22} - \phi_{33}) + \phi_{22}(\phi_{22} - 12\phi_{23} - 19\phi_{33}) + 28\phi_{23}^2$$
$$+ 2\phi_{33}(5\phi_{23} - 6\phi_{33}) + 2\big[\phi_{23}(2\phi_{33} + \phi_{23}) - \phi_{33}(2\phi_{33} + \phi_{22})\big]R\ ;$$
$$d_1 = 2\big[\phi_{11}(3\phi_{23}^2 - \phi_{22}\phi_{23} - \phi_{22}\phi_{33} - \phi_{23}\phi_{33}) + \phi_{23}(8\phi_{23}^2 - 3\phi_{23}\phi_{33} - 2\phi_{33}^2)$$
$$+ \phi_{22}(\phi_{22}\phi_{23} + 4\phi_{22}\phi_{33} + 7\phi_{33}^2 - 9\phi_{23}\phi_{33} - 6\phi_{23}^2)$$
$$+ (\phi_{22}\phi_{33}^2 + \phi_{23}^3 - \phi_{23}^2\phi_{33} - \phi_{22}\phi_{23}\phi_{33})R\big]\ ;$$
$$e_1 = \phi_{11}\phi_{22}(\phi_{33}^2 + \phi_{22}\phi_{33} - 2\phi_{23}\phi_{33} - \phi_{23}^2) + \phi_{23}\phi_{33}(4\phi_{22}^2 - \phi_{11}\phi_{23} - 2\phi_{23}^2)$$
$$+ \phi_{22}\phi_{23}(\phi_{23}\phi_{33} + 2\phi_{33}^2 + \phi_{22}\phi_{23} - 4\phi_{23}^2) - \phi_{22}^2\phi_{33}(3\phi_{33} + \phi_{22})$$
$$+ 2\phi_{23}^3(\phi_{11} + \phi_{23})\ ;$$
$$p_1 = 32\ ;$$
$$q_1 = 8(\phi_{11} + 8\phi_{23} - 4\phi_{33} - 5\phi_{22})\ ;$$
$$r_1 = 8(\phi_{33} + 2\phi_{22} - 2\phi_{23} - \phi_{11})(\phi_{22} + \phi_{33} - 2\phi_{23})\ ;$$
$$s_1 = 2(\phi_{11} - \phi_{22})(\phi_{22} + \phi_{33} - 2\phi_{23})^2\ ;$$
$$t_1 = 0$$

$$a_2 = 2\big[\phi_{11} - \phi_{22} + 2(\phi_{23} + \phi_{33}) - 2(\phi_{23} + \phi_{33})R\big]\ ;$$
$$b_2 = \phi_{11}(6\phi_{23} - \phi_{22} - \phi_{33}) + \phi_{22}(\phi_{22} - 6\phi_{23} - 11\phi_{33}) + 8\phi_{33}^2 + 4\phi_{23}^2$$
$$+ 4(\phi_{33}^2 + \phi_{22}\phi_{33} - 2\phi_{23}^2)R\ ;$$
$$c_2 = 2\big[\phi_{11}(3\phi_{23}^2 - \phi_{22}\phi_{23} - \phi_{22}\phi_{33} - \phi_{23}\phi_{33}) + \phi_{22}(\phi_{22}\phi_{23} + 3\phi_{22}\phi_{33}$$
$$- 3\phi_{23}^2 - 3\phi_{23}\phi_{33}) + 2\phi_{23}(\phi_{23}\phi_{33} + 3\phi_{33}^2 - \phi_{23}^2) - 4\phi_{33}^3$$
$$+ 2(\phi_{33} - \phi_{23})(\phi_{23}^2 - \phi_{22}\phi_{33})R\big]\ ;$$
$$d_2 = (\phi_{22}\phi_{33} - \phi_{23}^2)$$
$$\times\ \big[4(\phi_{23} - \phi_{33})^2 + \phi_{22}(2\phi_{23} - \phi_{33}) + \phi_{11}(\phi_{22} - 2\phi_{23} + \phi_{33}) - \phi_{22}^2\big]\ ;$$
$$e_2 = 0\ ;$$
$$p_2 = \phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})\ ;$$
$$q_2 = 4\phi_{23}(\phi_{11} + 4\phi_{23} - \phi_{22}) + 8(3\phi_{23} - \phi_{22})\phi_{33} + 16\phi_{33}^2\ ;$$
$$r_2 = 2\big[3\phi_{23}^2(\phi_{11} + 4\phi_{23} - \phi_{22}) + \phi_{33}(\phi_{22}^2 - \phi_{11}\phi_{22} + 16\phi_{23}^2 - 12\phi_{22}\phi_{23})$$
$$- 8\phi_{33}^2(\phi_{22} + \phi_{33})\big]\ ;$$
$$s_2 = 4(\phi_{23}^2 - \phi_{22}\phi_{33})\big[\phi_{23}(\phi_{11} + 4\phi_{23} - \phi_{22}) + 2\phi_{33}(\phi_{23} - \phi_{22}) - 4\phi_{33}^2\big]\ ;$$
$$t_2 = (\phi_{11} - \phi_{22} + 4\phi_{23} - 4\phi_{33})(\phi_{23}^2 - \phi_{22}\phi_{33})^2$$

$$a_3 = 2(R-1)\ ;$$
$$b_3 = \phi_{11} + \phi_{22} - 2\phi_{23}\ ;$$
$$c_3 = d_3 = e_3 = 0\ ;$$
$$p_3 = 8\ ;$$
$$q_3 = 2(\phi_{11} - \phi_{22})\ ;$$
$$r_3 = s_3 = t_3 = 0$$

$$a_4 = 2\big[\phi_{22} - \phi_{11} - 2(\phi_{23} + \phi_{33}) + 2(\phi_{23} + \phi_{33})R\big]\ ;$$
$$b_4 = 2\big[\phi_{11}(\phi_{22} - 2\phi_{23} + \phi_{33}) + \phi_{22}(4\phi_{23} - \phi_{22}) + \phi_{33}(7\phi_{22} + 2\phi_{23} - 4\phi_{33}) + 2(\phi_{23}^2 - 2\phi_{33}^2 - \phi_{33}(\phi_{22} + 2\phi_{23}))R\big]\ ;$$
$$c_4 = 4\big[\phi_{11}\phi_{23}(\phi_{22} + \phi_{33}) + \phi_{22}(3\phi_{23}^2 + 3\phi_{23}\phi_{33} + 3\phi_{33}^2) - \phi_{22}^2(\phi_{23} + 3\phi_{33})$$
$$+ \phi_{23}(2\phi_{23}^2 - 3\phi_{23}\phi_{33} - 6\phi_{33}^2) + (\phi_{22}\phi_{33}(\phi_{23} + 3\phi_{33}) - \phi_{23}^2(\phi_{23} + 3\phi_{33}))R\big]\ ;$$
$$d_4 = 2\big[(\phi_{22}\phi_{33} - \phi_{23}^2)(\phi_{22}^2 - \phi_{11}(\phi_{22} + 2\phi_{23} + \phi_{33}) + 10\phi_{23}\phi_{33} - \phi_{22}(4\phi_{23} + 3\phi_{33}) + 2(\phi_{23}^2 - \phi_{22}\phi_{33})R)\big]\ ;$$
$$e_4 = 2(\phi_{11} + \phi_{22} - 2\phi_{23})(\phi_{23}^2 - \phi_{22}\phi_{33})^2\ ;$$
$$p_4 = \phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})\ ;$$
$$q_4 = 4(\phi_{23}(\phi_{11} - \phi_{22} + 4\phi_{23}) - 2(\phi_{22} - 3\phi_{23})\phi_{33} + 4\phi_{33}^2)\ ;$$
$$r_4 = 2\big[3\phi_{23}^2(\phi_{11} + 4\phi_{23} - \phi_{22}) + \phi_{33}(16\phi_{23}^2 + \phi_{22}^2) - \phi_{22}\phi_{33}(12\phi_{23} + \phi_{11}) - 8\phi_{33}^2(\phi_{22} + \phi_{33})\big]\ ;$$
$$s_4 = 4(\phi_{23}^2 - \phi_{22}\phi_{33})\big[\phi_{23}(\phi_{11} - \phi_{22} + 4\phi_{23}) + 2\phi_{33}(\phi_{23} - \phi_{22}) - 4\phi_{33}^2\big]\ ;$$
$$t_4 = (\phi_{11} - \phi_{22} + 4\phi_{23} - 4\phi_{33})(\phi_{23}^2 - \phi_{22}\phi_{33})^2$$

$$R = \sqrt{1 - 2g(\phi_{11} + \phi_{23} - 2\phi_{33}) + g^2(\phi_{23}^2 - \phi_{22}\phi_{33} + \phi_{11}\phi_{22} - 2\phi_{11}\phi_{23} + \phi_{11}\phi_{33})}$$

3f. Toy Problem 2.

Nonlinear solution – Tight-binding interaction in a chain Expansion in powers of g

$$V_1^{\mathtt{inv}} = \frac{\phi_{11} - \phi_{22}}{8} + \frac{\phi_{33} + \phi_{23}}{2} + O(g)$$

$$V_3^{\mathtt{inv}} = \frac{\phi_{22} - \phi_{11}}{8} + \frac{\phi_{33} - \phi_{23}}{2} + O(g)$$

$$V_2^{\mathtt{inv}} = \frac{2}{g}\frac{\phi_{11} - \phi_{22}}{\phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})} + \frac{\phi_{11} - \phi_{22} + 4(\phi_{23} - \phi_{33})}{\big(\phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})\big)^2}\Big[\phi_{22}^2 + \phi_{22}\big(2\phi_{23} + \phi_{33}\big) - \phi_{11}\big(\phi_{22} - 2\phi_{23} - 3\phi_{33}\big) - 4\phi_{33}\big(\phi_{23} + \phi_{33}\big)\Big] + O(g^1)$$

$$V_4^{\mathtt{inv}} = \frac{2}{g}\frac{\phi_{22} - \phi_{11}}{\phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})} + \frac{2}{\big(\phi_{11} - \phi_{22} + 4(\phi_{23} + \phi_{33})\big)^2} \times \Big[\phi_{22}(\phi_{11} - \phi_{22})(\phi_{11} - \phi_{22} + 4\phi_{23}) + 4\phi_{23}\phi_{33} \times (\phi_{11} + 3\phi_{22}) + 8\phi_{33}^2\big(\phi_{11} + \phi_{22} - 2(\phi_{23} + \phi_{33})\big) - \phi_{33}(\phi_{11} - \phi_{22})^2\Big] + O(g^1)$$

Now the solution
is unique if g=/=0

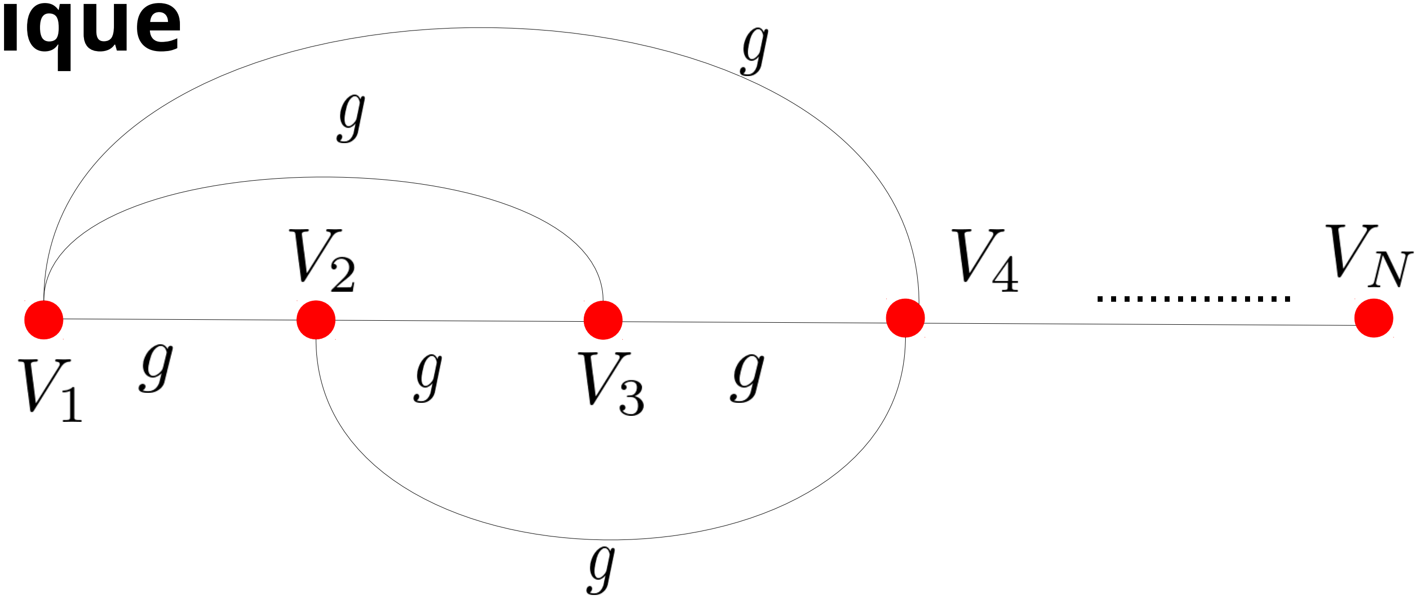
4a. Toy Problem 3.

Linear chain with N degrees of freedom;

Interaction on a fully-connected graph;

The "chain" geometry is only important for measurement matrix definition;

Inverse problem on a fully connected graph can be analytically solved if the solution is unique



4b. Toy Problem 3.

$$G_{nn'} = g(1 - \delta_{nn'})$$

$$T_{nn'} = \kappa_n \delta_{nn'} + g\frac{\kappa_n \kappa_{n'}}{1 - gS} \; , \quad 1 \leq n, n' \leq N$$

$$\kappa_n = \frac{V_n}{1 + gV_n} \; , \quad S = \sum_{n=1}^{N} \kappa_n$$

$$A_{ln} = e^{-\mathrm{i}\frac{2\pi}{N} ln} \; , \quad B_{nl} = e^{\mathrm{i}\frac{2\pi}{N} nl} \; , \quad -L \leq l \leq L$$

We send plane waves with the wave number $k = 2\pi L/Nh$ at different angles to the chain. Here h is the spacing in the chain. The wavelength is $\lambda = Nh/L$.

4c. Toy Problem 3. Linear Regime g=0.

$$T_{nn'} = V_n \delta_{nn'}$$

$$\sum_{n=1}^{N} A_{ln} V_n B_{nm} = \phi_{lm} \ , \quad -L \leq l, m \leq L$$

$$\sum_{n=1}^{N} e^{-\mathrm{i}\frac{2\pi}{N} ln} V_n e^{\mathrm{i}\frac{2\pi}{N} mn} = \phi_{lm}$$

$(2L+1)^2$ equations, but not all are independent

$$\tilde{V}_{m-l} = \phi_{lm} \qquad \text{where} \qquad \tilde{V}_m = \sum_{n=1}^{N} V_n e^{\mathrm{i}\frac{2\pi}{N} nm}$$

So, we know from data all discrete Fourier coefficients of V_n in the band $-2L \leq m \leq 2L$. If $2L \geq M = (N-1)/2$ (assuming N is odd), then we know all DFT coefficients and can invert the DFT uniquely. If $2L < M$, some DFT coefficients of V_n are fundamentally unknown. The minimum-L_2 norm solution is low-band pass-diltered version of V_n:

Minimum norm inverse solution →

$$V_n^{\mathtt{inv}} = \frac{1}{N} \sum_{m=-2L}^{2L} \tilde{V}_m e^{-\mathrm{i}\frac{2\pi}{N} nm}$$

4d. Toy Problem 3. Nonlinear Regime $g=/=0$.

$$\tilde{\kappa}_m^{\texttt{inv}} = \begin{cases} \dfrac{\phi_{0m}}{1+g\phi_{00}} & , \quad -L \le m \le L \\ \phi_{-L,m-L} - g\dfrac{\phi_{0L}\phi_{0,m-L}}{1+g\phi_{00}} & , \quad L < m \le 2L \\ \phi_{L,m+L} - g\dfrac{\phi_{0,-L}\phi_{0,m+L}}{1+g\phi_{00}} & , \quad -2L \le m < -L \\ \text{Unknown} & , \quad m > |2L| \end{cases}$$

Here $\tilde{\kappa}_m = \sum_{n=1}^{N} \kappa_n e^{\mathrm{i}\frac{2\pi}{N}nm}$, $\kappa_n = \dfrac{V_n}{1+gV_n}$

If all $\tilde{\kappa}_m$ are known with $|m| \le M = (N-1)/2$

$$\kappa_n^{\texttt{inv}} = \frac{1}{N}\sum_{m=-M}^{M} \tilde{\kappa}_m e^{-\mathrm{i}\frac{2\pi}{N}nm} , \quad V_n^{\texttt{inv}} = \frac{\kappa_n^{\texttt{inv}}}{1-g\kappa_n^{\texttt{inv}}}$$

4d. Toy Problem 3. Nonlinear Regime $g =/= 0$ (cont.)

If NOT all $\tilde{\kappa}_m$ are known with $|m| \leq M = (N-1)/2$
we must make some guess about the coefficients that are unknown:
a) Zero
b) Random
c) See solution so the norm of V is minimized (more difficult)

If we set the unknown DFT coefficients to 0, then

$$\kappa_n^{\texttt{inv}} = \frac{1}{N} \sum_{m=-2L}^{2L} \tilde{\kappa}_m e^{-\mathrm{i}\frac{2\pi}{N} nm} \ , \quad V_n^{\texttt{inv}} = \frac{\kappa_n^{\texttt{inv}}}{1 - g\kappa_n^{\texttt{inv}}}$$

If we do this, here is what would happen:

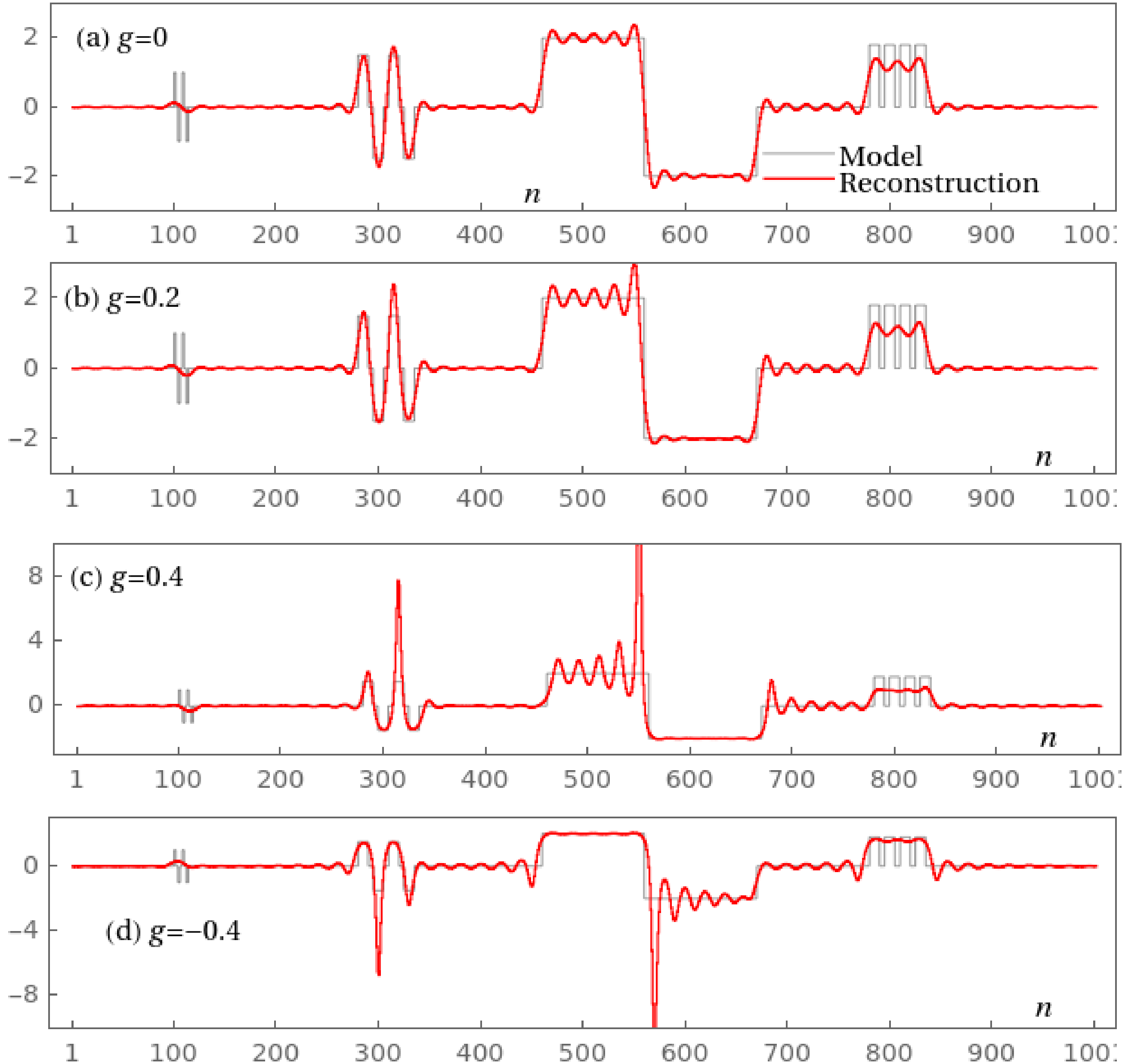
$$N = 1,001$$

$$M = \frac{N-1}{2} = 500$$

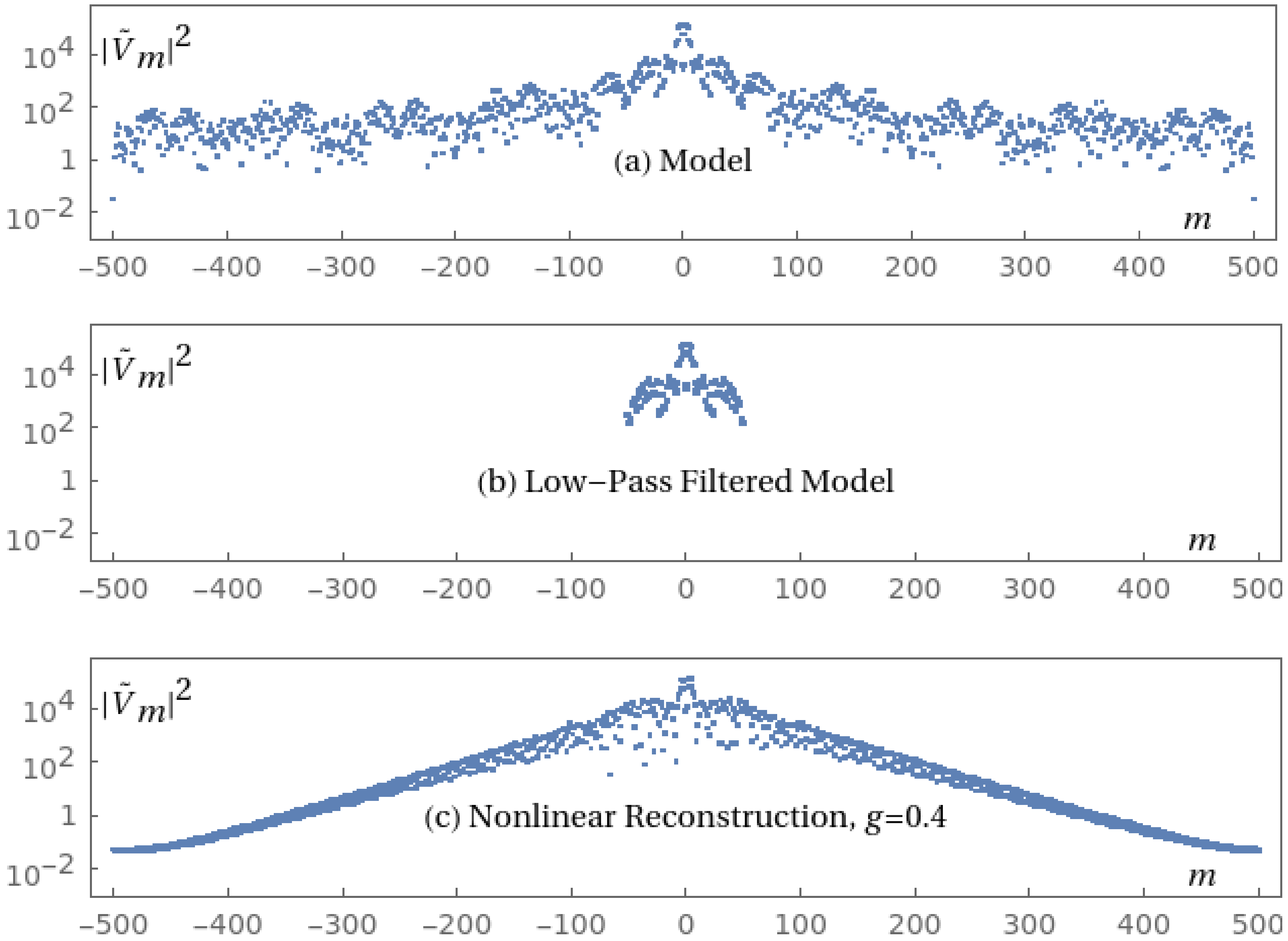
$$L = 25$$

$$M/2L = 10$$

Only about 1/10 of DFT coefficients are known (linear problem is strongly band-limited)

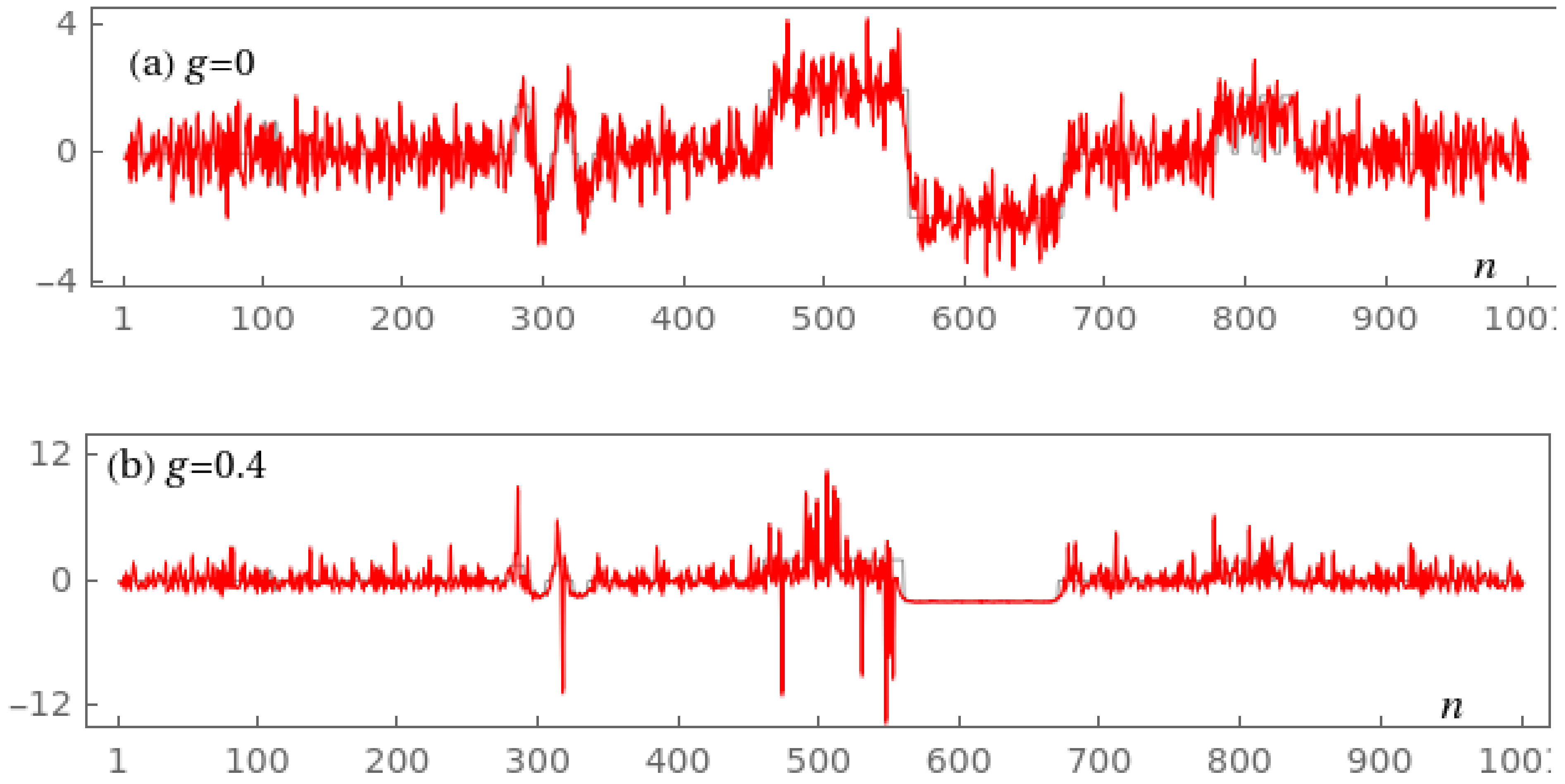


Unknown coefficients set to 0
Not the minimum norm of V solution.



$|\tilde{V}_m|^2$
10^4
10^2
1
10^{-2}
(a) Model
m
−500 −400 −300 −200 −100 0 100 200 300 400 500
$|\tilde{V}_m|^2$
10^4
10^2
1
10^{-2}
(b) Low−Pass Filtered Model
m
−500 −400 −300 −200 −100 0 100 200 300 400 500
$|\tilde{V}_m|^2$
10^4
10^2
1
10^{-2}
(c) Nonlinear Reconstruction, g=0.4
m
−500 −400 −300 −200 −100 0 100 200 300 400 500

Toy Problem 3: This is what happens if we fill the unknown coefficients with random values, which does not make sense but we tried it anyway
2L/M=0.1

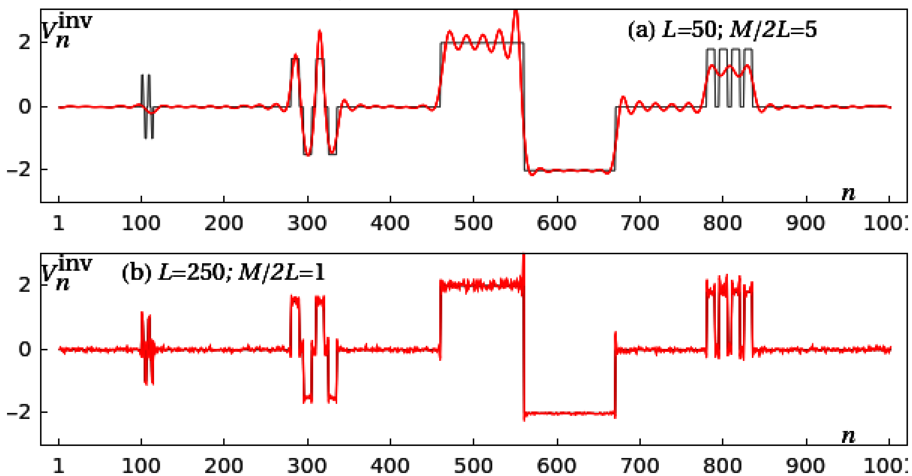


TOY PROBLEM 3: Reconstructions with noise in the known coefficients (unknown are set to 0)

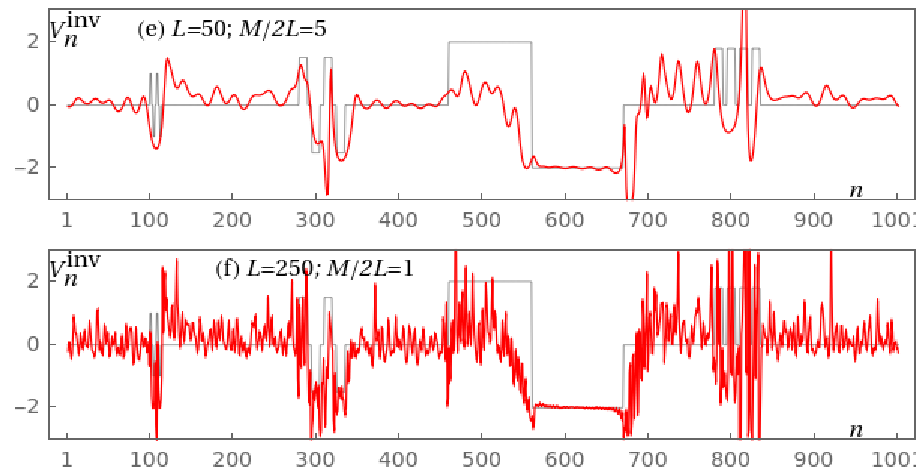
$$\phi_{jl} \longrightarrow \phi_{jl} + RZ\sqrt{\langle|\tilde{\kappa}|^2\rangle} \qquad P[|Z|] = \sqrt{2/\pi}\,\exp(-|Z|^2/2)$$

Phase of Z is random

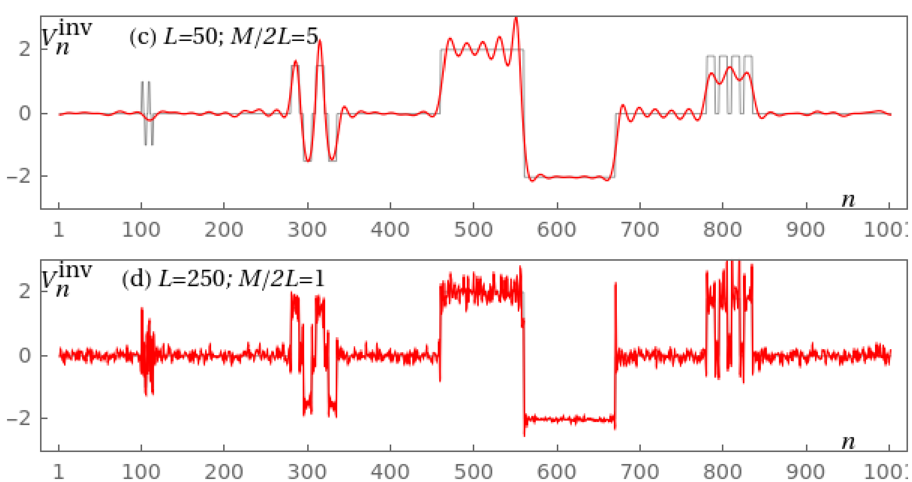
g=0.2,R=0.1



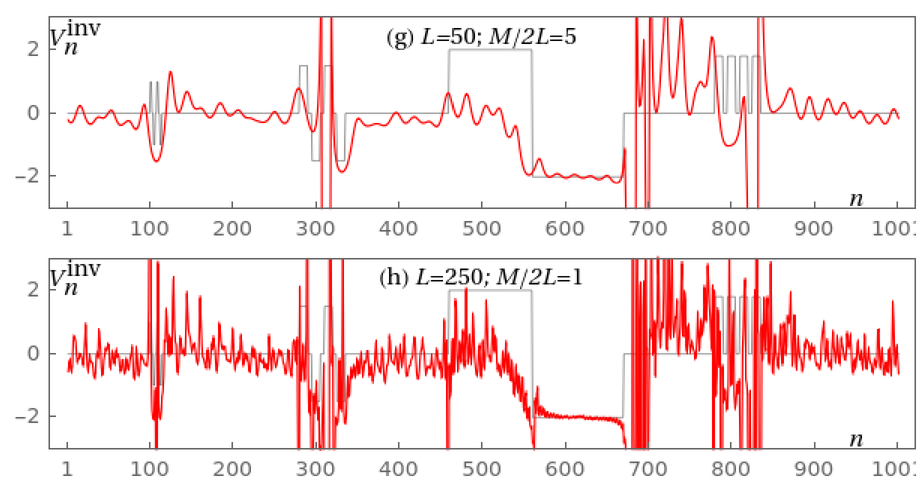
g=0.4,R=0.1



g=0.2,R=0.2



g=0.4,R=0.2



5a. Distorted Born Approximation and Tangent Spaces

$$\mathtt{U} = \mathtt{D} + \mathtt{V}$$

Small deviation from initial guess (V)

Some initial guess (D)

Total potential (U)

$$\mathtt{T}[\mathtt{U}] = \mathtt{T}[\mathtt{D} + \mathtt{V}] \approx \mathtt{T}[\mathtt{D}] + \mathtt{S}[\mathtt{D}]\,\mathtt{V}\,\mathtt{S}^{\mathtt{T}}[\mathtt{D}]$$

where

$$\mathtt{S}[\mathtt{X}] = (\mathtt{I} - \mathtt{XG})^{-1}\ , \quad \mathtt{S}^{\mathtt{T}}[\mathtt{X}] = (\mathtt{I} - \mathtt{GX})^{-1}$$

$$(\mathtt{A}\mathtt{S}[\mathtt{D}])\,\mathtt{V}\,(\mathtt{S}^{\mathtt{T}}[\mathtt{D}]\mathtt{B}) = \Phi - \mathtt{A}\,\mathtt{T}[\mathtt{D}]\,\mathtt{B} \equiv \Psi[\mathtt{D}]$$

New data matrix (known) ($\Psi[\mathtt{D}]$)

Linear equation for V

5b. Distorted Born Approximation

$$U = D + V$$

Total potential (U) — Some initial guess (D) — Small deviation from initial guess (V)

$$T[U] = T[D + V] \approx T[D] + S[D]\, V\, S^{T}[D]$$

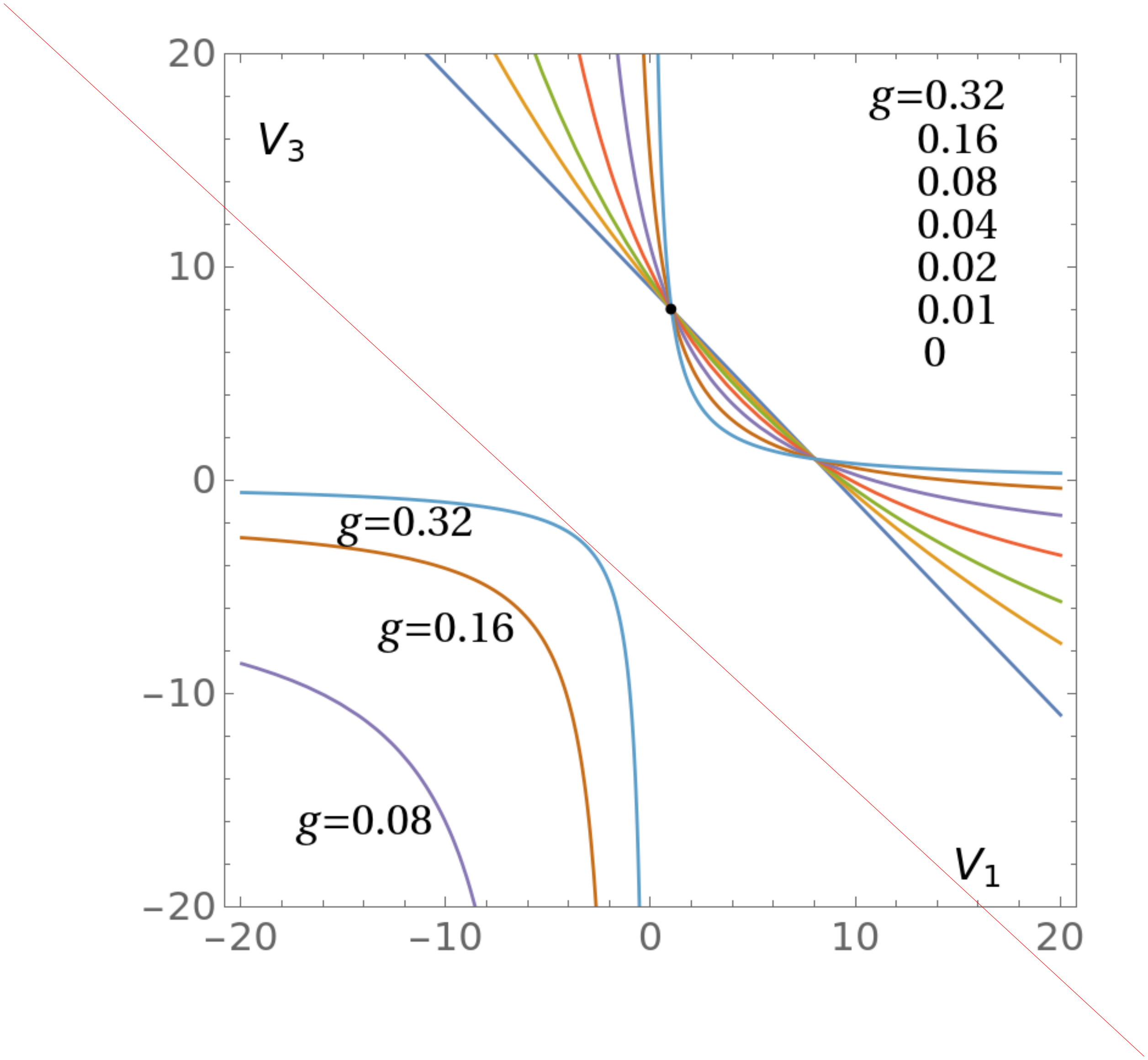
where

$$S[X] = (I - XG)^{-1}\ , \quad S^{T}[X] = (I - GX)^{-1}$$

$$(A S[D])\, V\, (S^{T}[D] B) = \Phi - A\, T[D]\, B \equiv \Psi[D]$$

New data matrix (known)

Linear equation for *V*



20
10
0
−10
−20
−20
−10
0
10
20
V_3
V_1
g=0.32
0.16
0.08
0.04
0.02
0.01
0
g=0.32
g=0.16
g=0.08

6a. Example 4 -- Realistic Interaction Chain with N=51x51 = 2601 voxels/particles

$$G_{nm} = g(1-\delta_{nm})\frac{e^{ik|\mathbf{r}_n-\mathbf{r}_m|}}{|\mathbf{r}_n-\mathbf{r}_m|/h}\ , \qquad k = \frac{\omega}{c}$$

Dimensionless parameter characterizing the strength of interaction (multiple scattering)

$$A_{ln} = e^{-\mathrm{i}(l/L)kz_n}\ , \quad B_{nl} = e^{\mathrm{i}(l/L)kz_n}$$

$$-L \le l \le L\ , \quad 1 \le n \le N\ , \quad z_n = nh$$

$(l/L)k$ ← projection of the incident (detected) wave vector onto the chain

$$\sum_n (\mathtt{AS}[\mathtt{D}])_{ln} V_n (\mathtt{S}^{\mathtt{T}}[\mathtt{D}]\mathtt{B})_{nm} = \psi_{lm}$$

(Distorted Born)

6b. Example 4: Pseudo-inverse for $\mathtt{V}$ using shifted Born

$$W_{nm}[\mathtt{D}] = \Big((\mathtt{AS}[\mathtt{D}])^*(\mathtt{AS}[\mathtt{D}])\Big)_{nm}\Big((\mathtt{S}^{\mathtt{T}}[\mathtt{D}]\mathtt{B})(\mathtt{S}^{\mathtt{T}}[\mathtt{D}]\mathtt{B})^*\Big)_{mn}$$

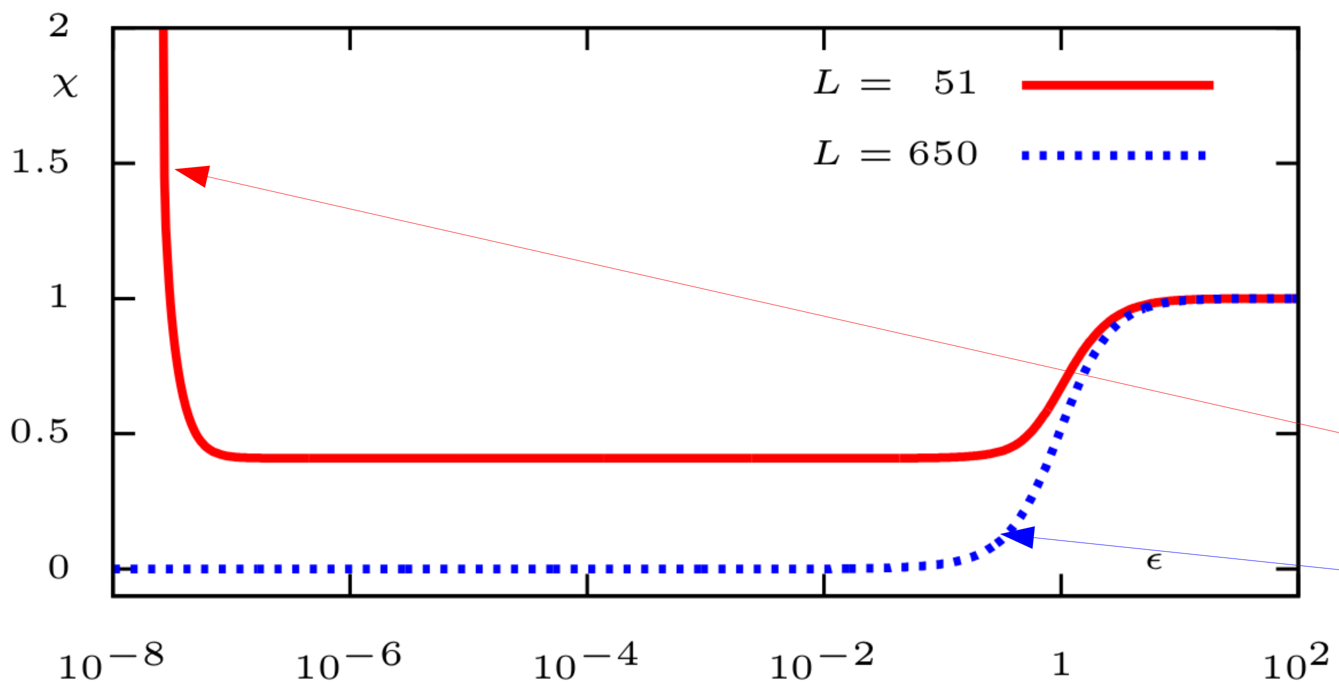
$\mathtt{v}$ is vector of diagonal elements of $\mathtt{V}$

$\mathtt{b}[\mathtt{D}]$ is data matrix $\Psi[\mathtt{D}]$ unlolled, i.e., $b_{(lm)} = \psi_{l,m}$

$$[\mathtt{W}[\mathtt{D}] + \lambda^2\mathtt{I}]\mathtt{v} = \mathtt{b}[\mathtt{D}]$$

Tikhonov regularization parameter

6c. Example 4: Pseudo-inverse at D=0, inverse crime



Error of the linearized pseudo-inverse of a model (sown below)
at D=0
with N=1,601

Inverse crime

Band-limited measurements

Non-band-limited measurements

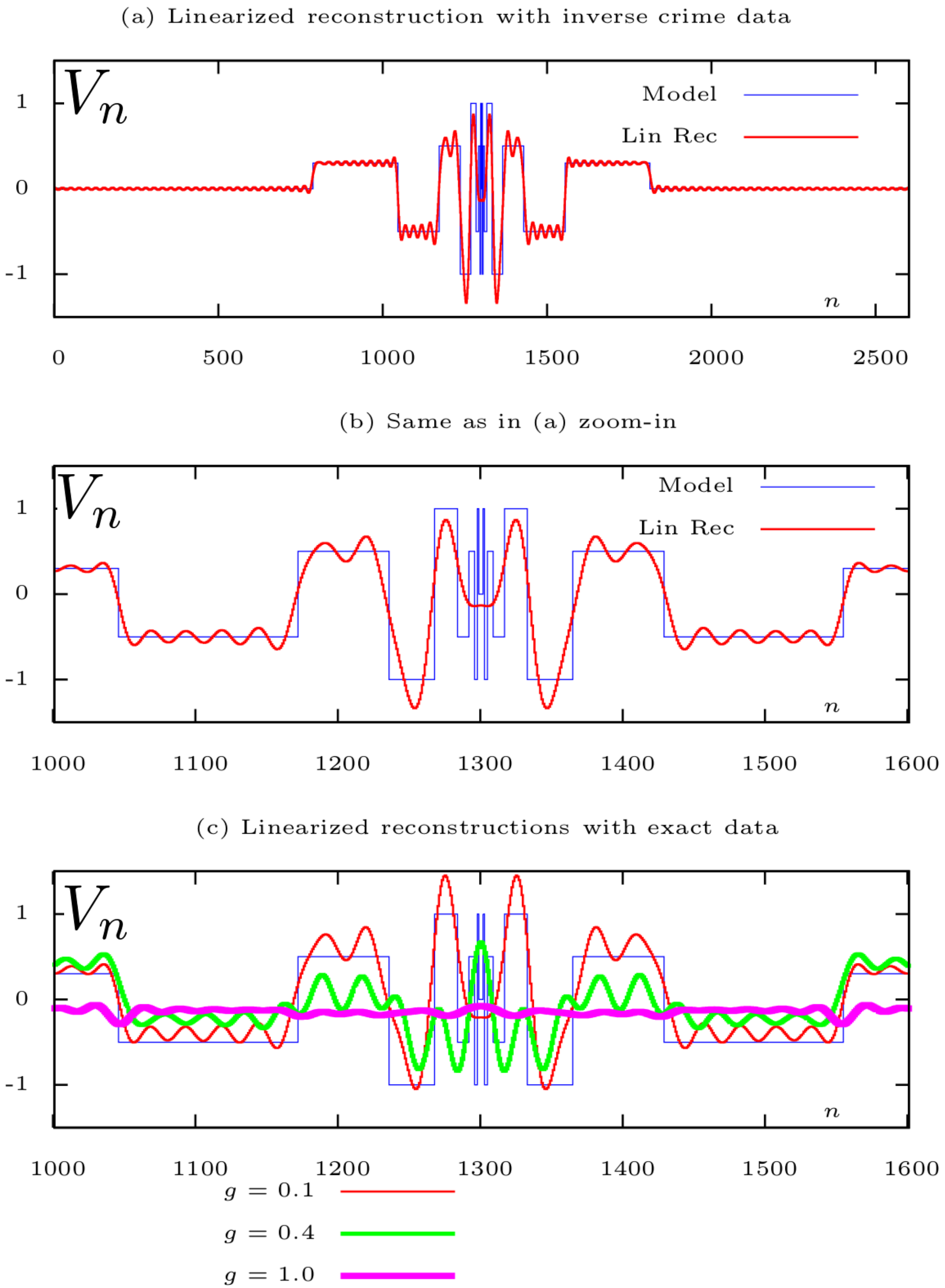
Here "chi" is shown where "chi-squared" is

$$\chi^2 = \frac{\sum_n |V_n^{\texttt{inv}} - V_n^{\texttt{mod}}|^2}{\sum_n |V_n^{\texttt{mod}}|^2}$$

6c. Example 4: Pseudo-inverse reconstructions with band-limited measurements

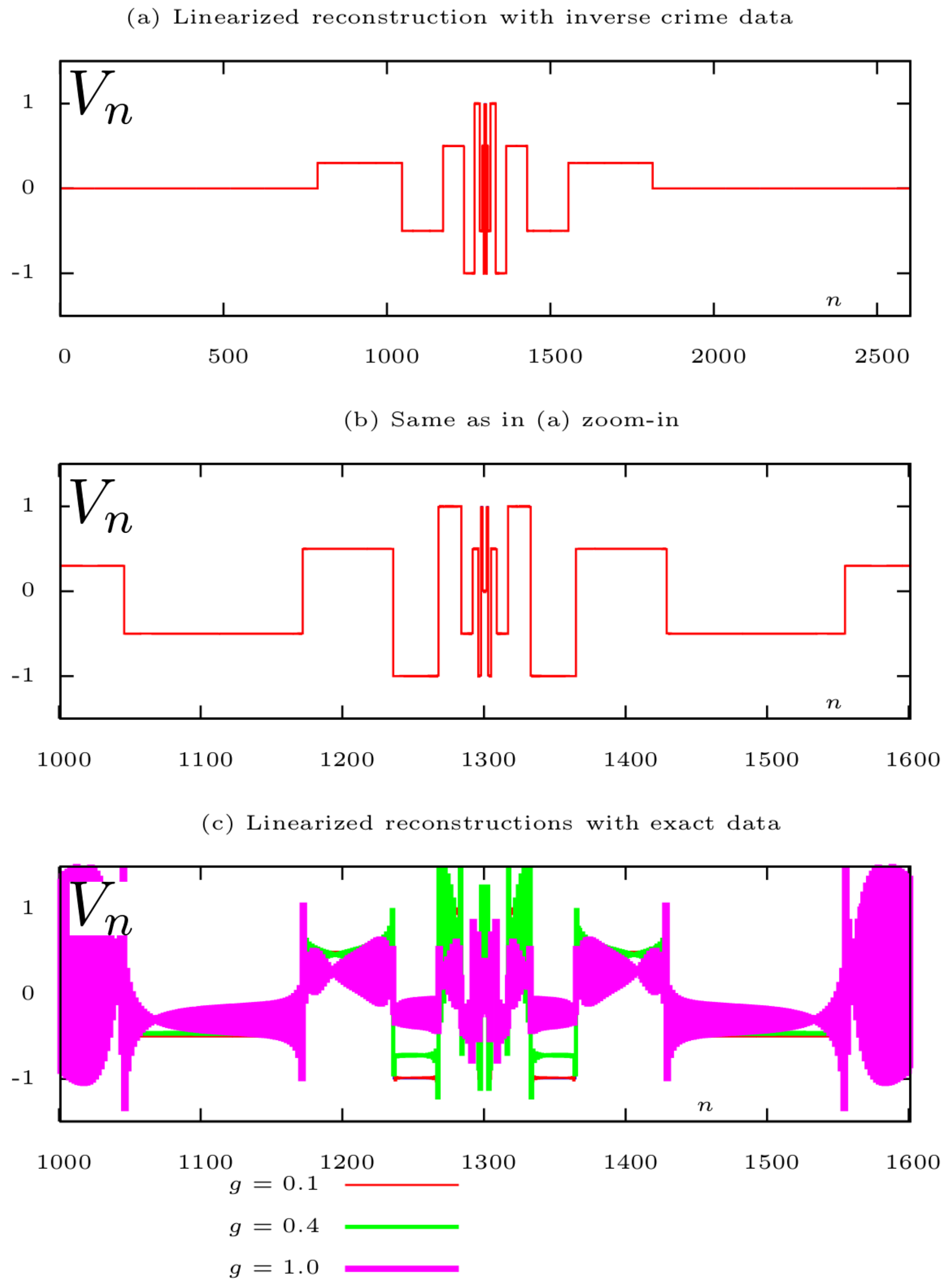
L=51

(inverse crime)

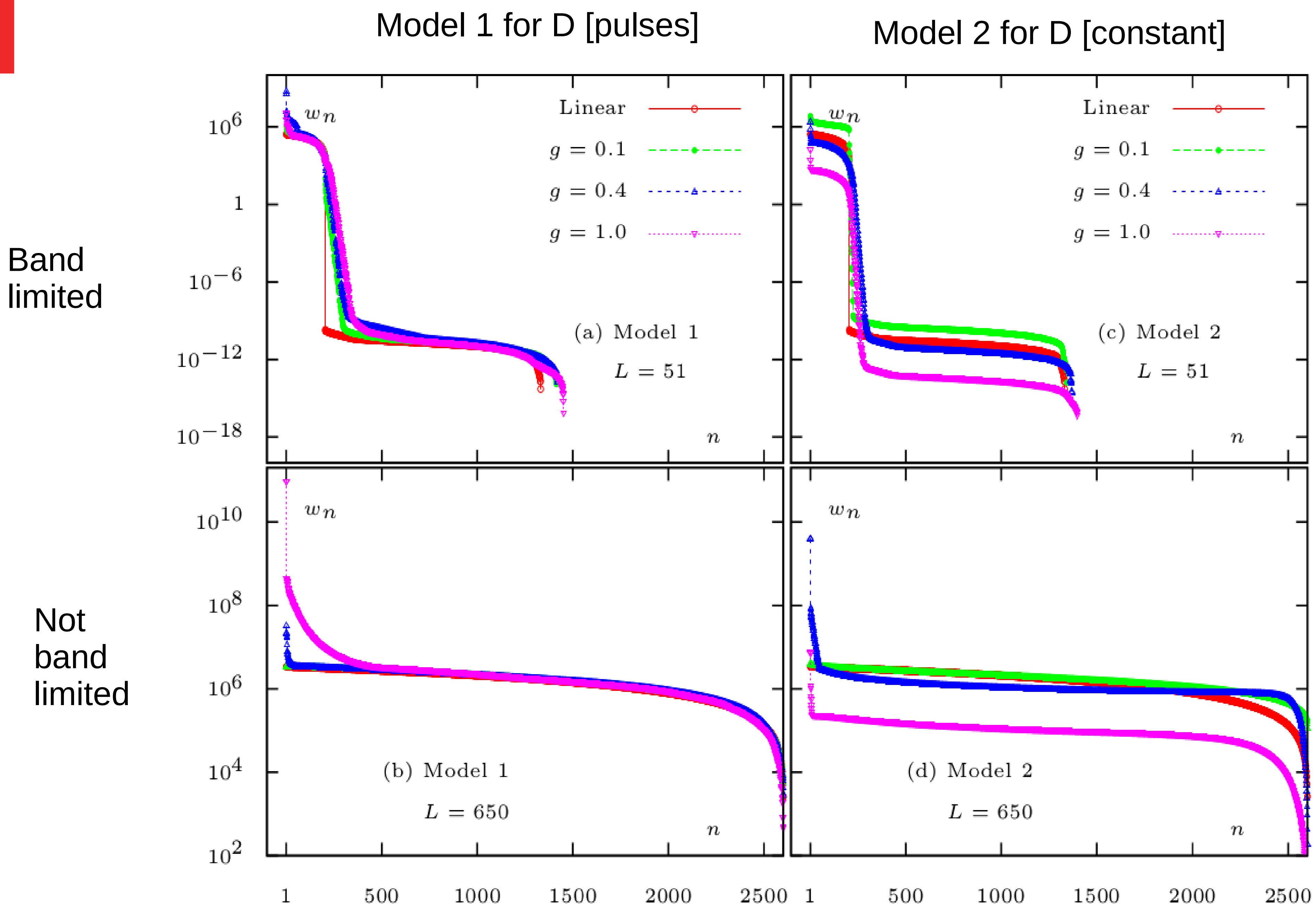


6d. Example 4: Pseudoinverse reconstructions with non-band-limited measurements

L=650



6e. Eigenvalues of W[D], size N=2,601

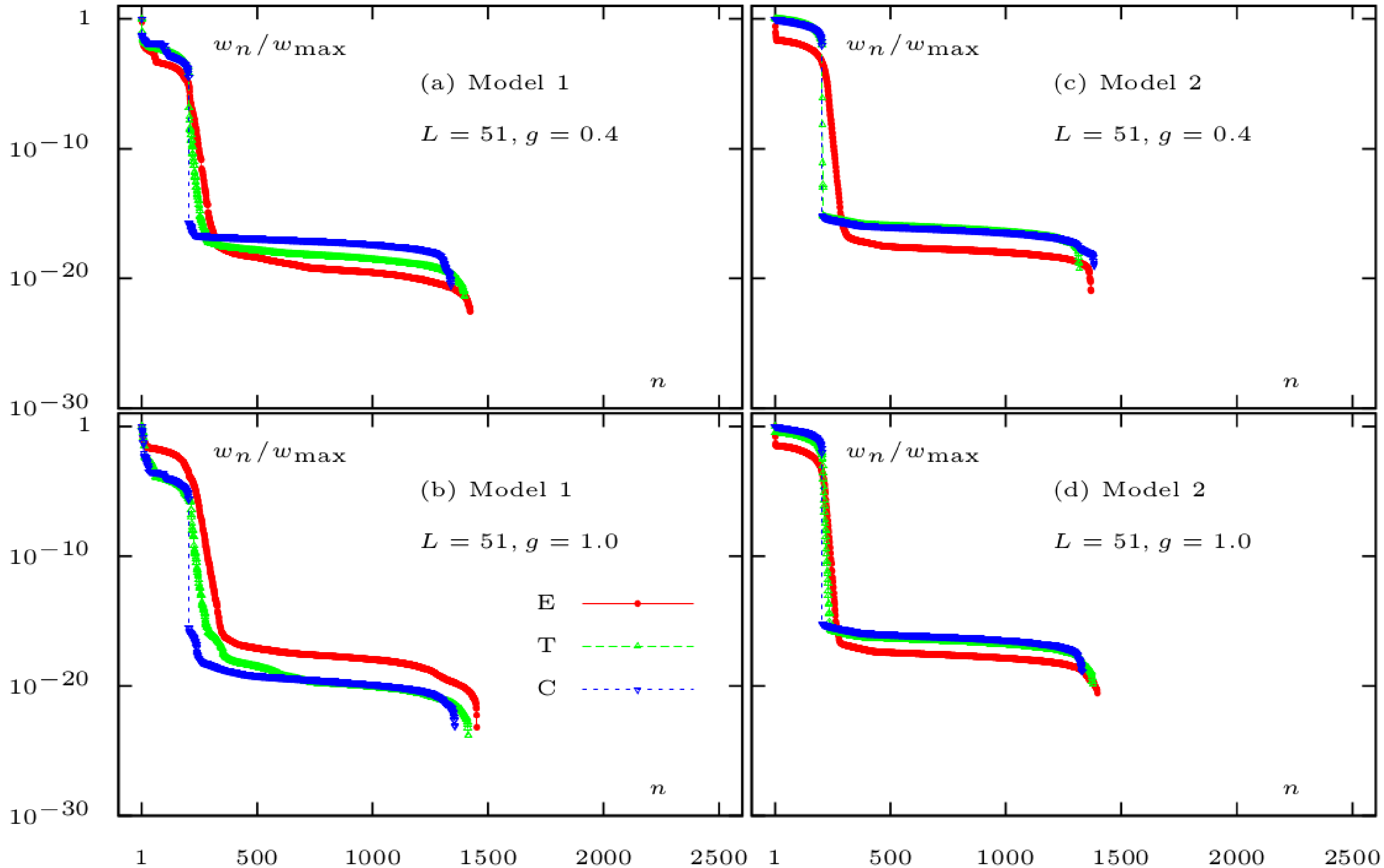


6f. Eigenvalues of W[D], different interactions

Model 1 for D [pulses] Model 2 for D [constant]

Band limited

Also band limited



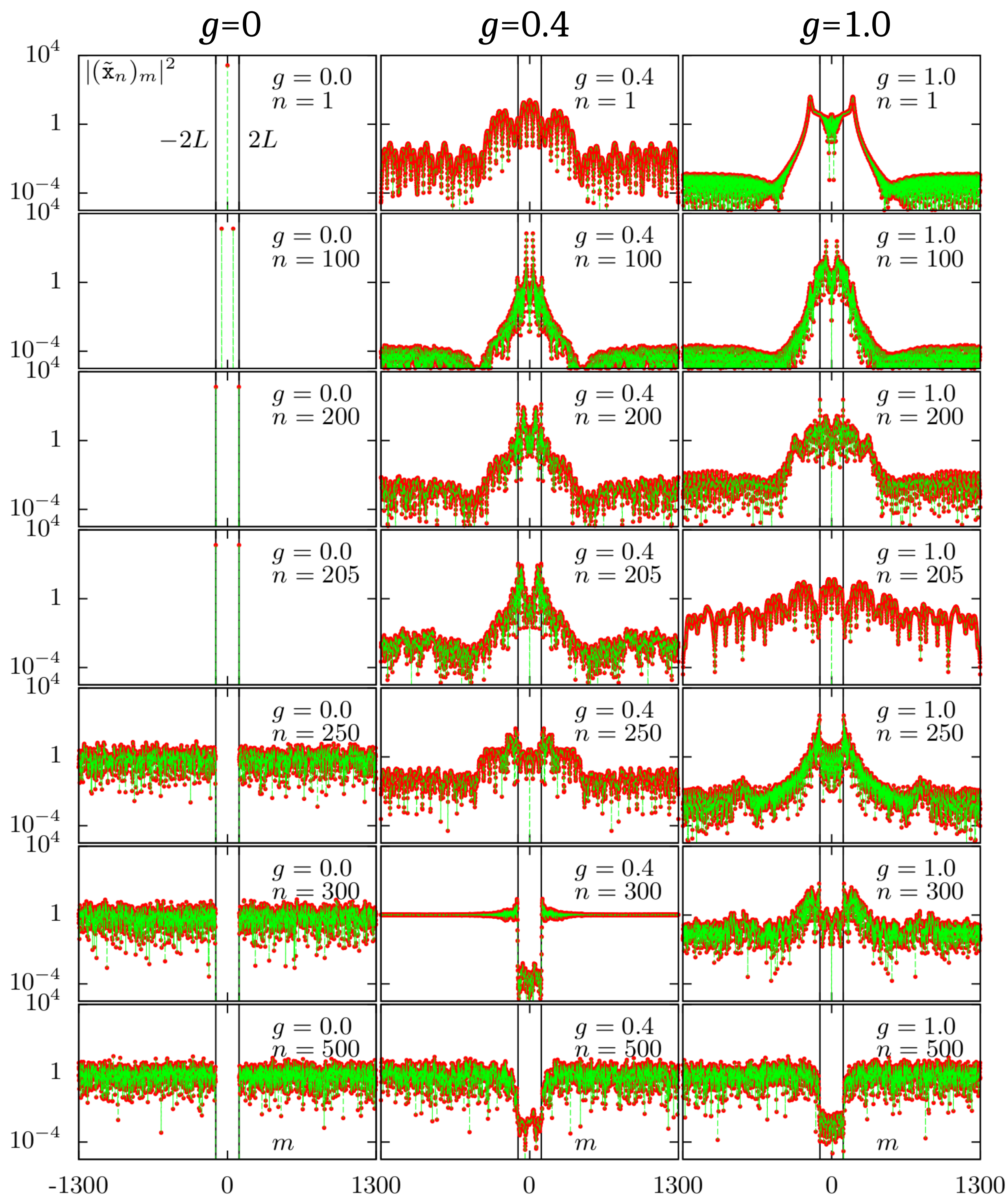
E-Exponential (realistic) T-Tight-binding C-Fully connected

6g. Eigenvalues of W[D]

L=51
Number of "significant" eigenvectors: 4*L*+1=205

Transition region: 205 to 350

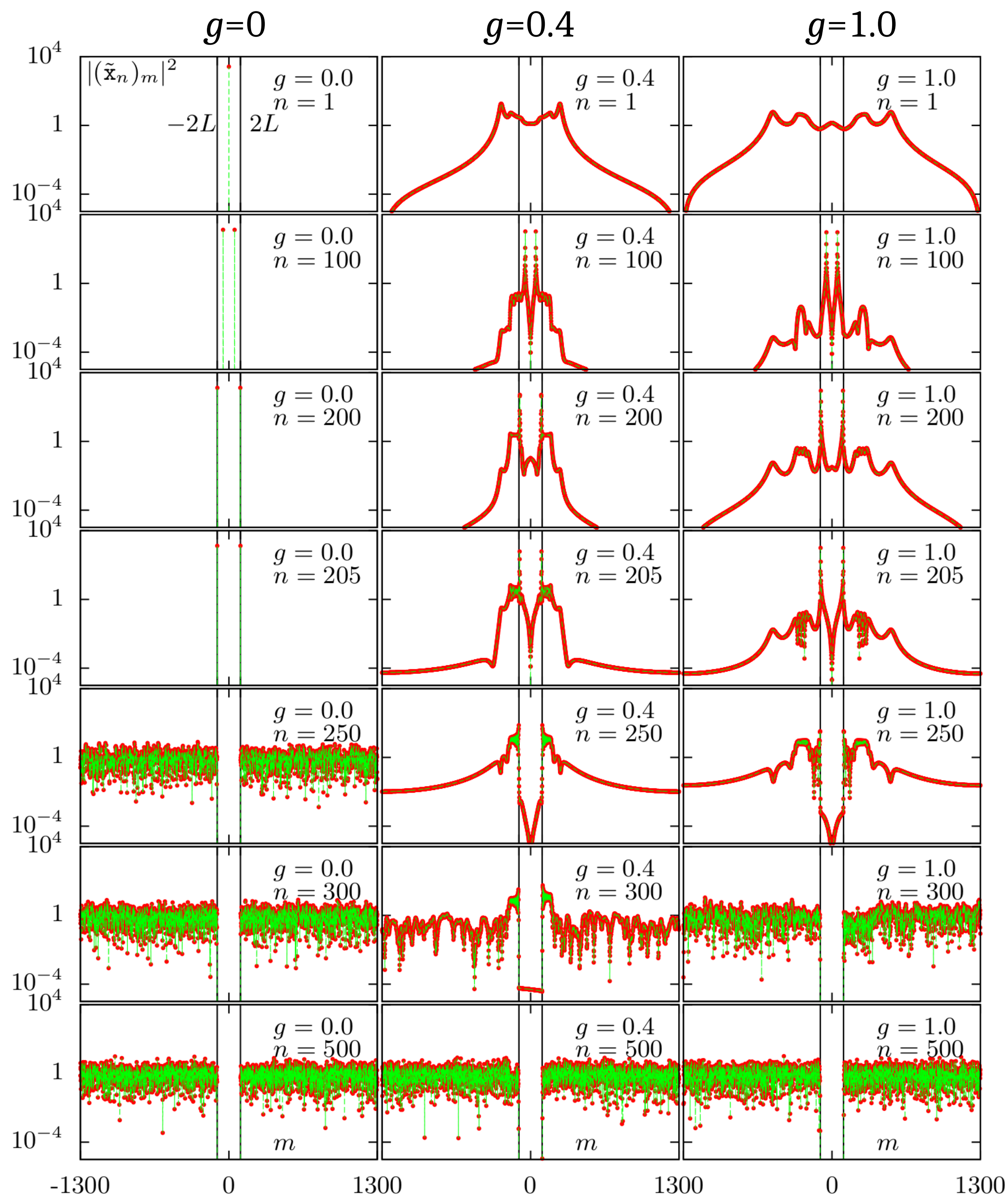
Model 1 for D



6h. Eigenvalues of W[D]

L=51
Number of "significant" eigenvectors: $4L+1=205$

Transition region: 205 to 350



CONCLUSIONS

- Nonlinearity of ISP is unlikely to help significantly with resolution. At least, it is an uphill struggle.
- On the other hand, nonlinearity can easily make the ISP ill-posed even if it is/qs well-posed in the linear regime.
- Perturbation of inverse solutions in the strength of interaction can be singular. This makes analysis difficult.
- Resolution limit exists

Several Lorentzians

$$L(x) = \frac{1}{\pi} \frac{\delta}{(x-a)^2 + \delta^2} \xrightarrow{\text{FT}} \tilde{L}(k) = e^{ika - |k|\delta}$$