An Extinction Paradox Involving Collimated Beams and its Resolution

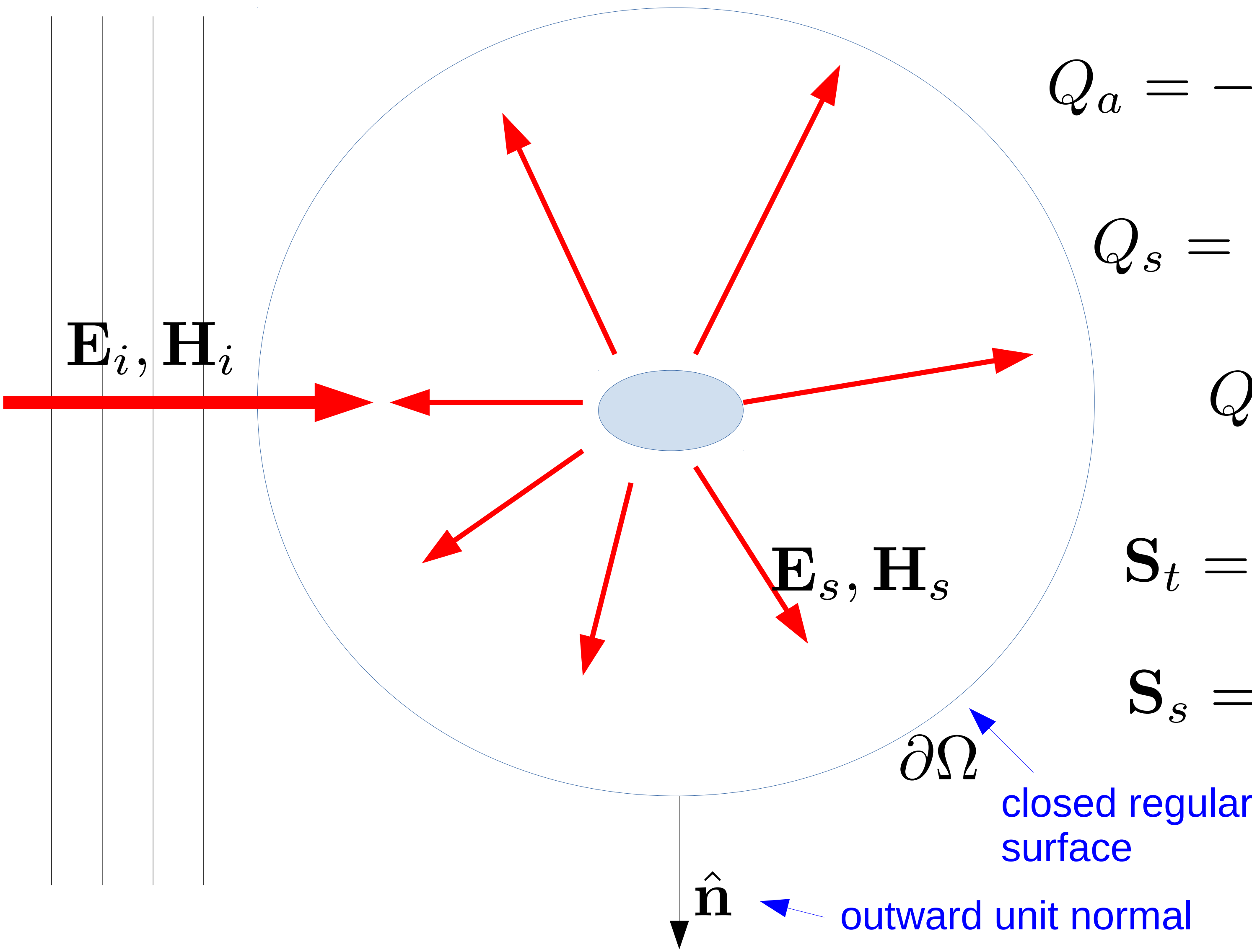
Vadim A. Makel

University of Pennsylvania
Department of Radiology
Philadelphia, PA

vmarkel@upenn.edu
http://whale.seas.upenn.edu/vmarkel

Some Definitions (single particle)

$$\mathbf{E}_t = \mathbf{E}_i + \mathbf{E}_s \;, \quad \mathbf{H}_t = \mathbf{H}_i + \mathbf{H}_s$$



$$Q_a = -\oint_{\partial\Omega} [\mathbf{S}_t(\mathbf{r}) \cdot \hat{\mathbf{n}}] d^2 r$$

$$Q_s = \oint_{\partial\Omega} [\mathbf{S}_s(\mathbf{r}) \cdot \hat{\mathbf{n}}] d^2 r$$

$$Q_e = Q_a + Q_s$$

$$\mathbf{S}_t = \frac{c}{4\pi} \langle \mathbf{E}_t \times \mathbf{H}_t \rangle_{\text{time}}$$

$$\mathbf{S}_s = \frac{c}{4\pi} \langle \mathbf{E}_s \times \mathbf{H}_s \rangle_{\text{time}}$$

Extinction is an interference effect

$$Q_e = -\frac{c}{4\pi}\oint_{\partial\Omega}[\langle \mathbf{E}_i \times \mathbf{H}_i + \mathbf{E}_i \times \mathbf{H}_s + \mathbf{E}_s \times \mathbf{H}_i \rangle_{\text{time}} \cdot \hat{\mathbf{n}}]d^2r$$

$$= -\frac{c}{4\pi}\oint_{\partial\Omega}[\langle \mathbf{E}_i \times \mathbf{H}_s + \mathbf{E}_s \times \mathbf{H}_i \rangle_{\text{time}} \cdot \hat{\mathbf{n}}]d^2r$$

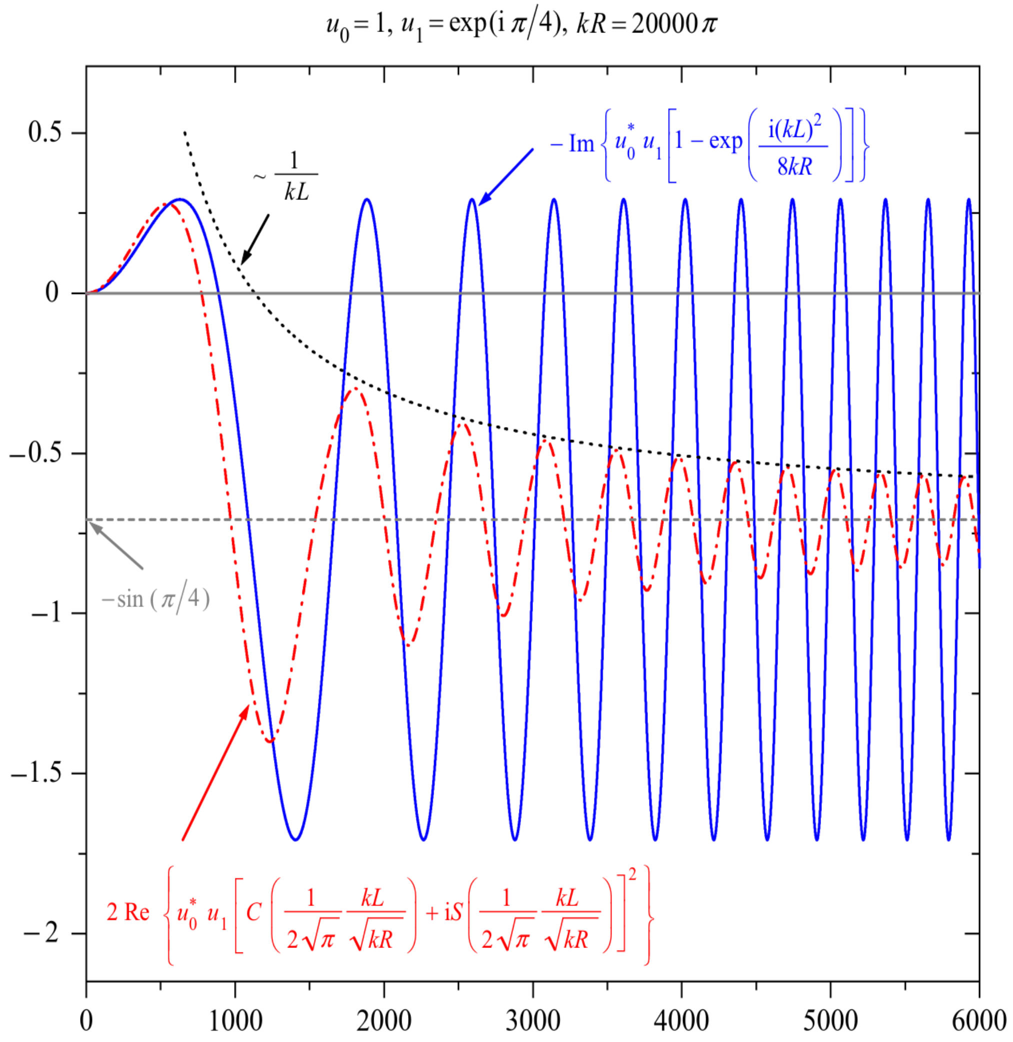
How do we measure extinction (how is it affecting measurements)?

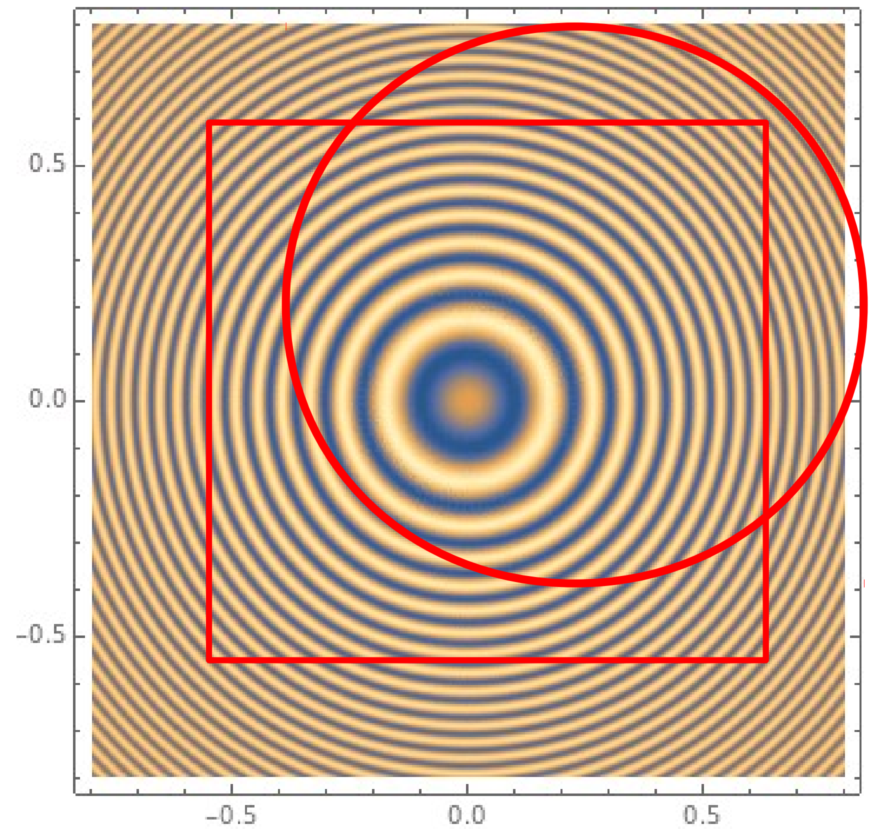
Historically, this question has led to various paradoxes...

Wrong explanation of the classical extinction paradox are still quite common...

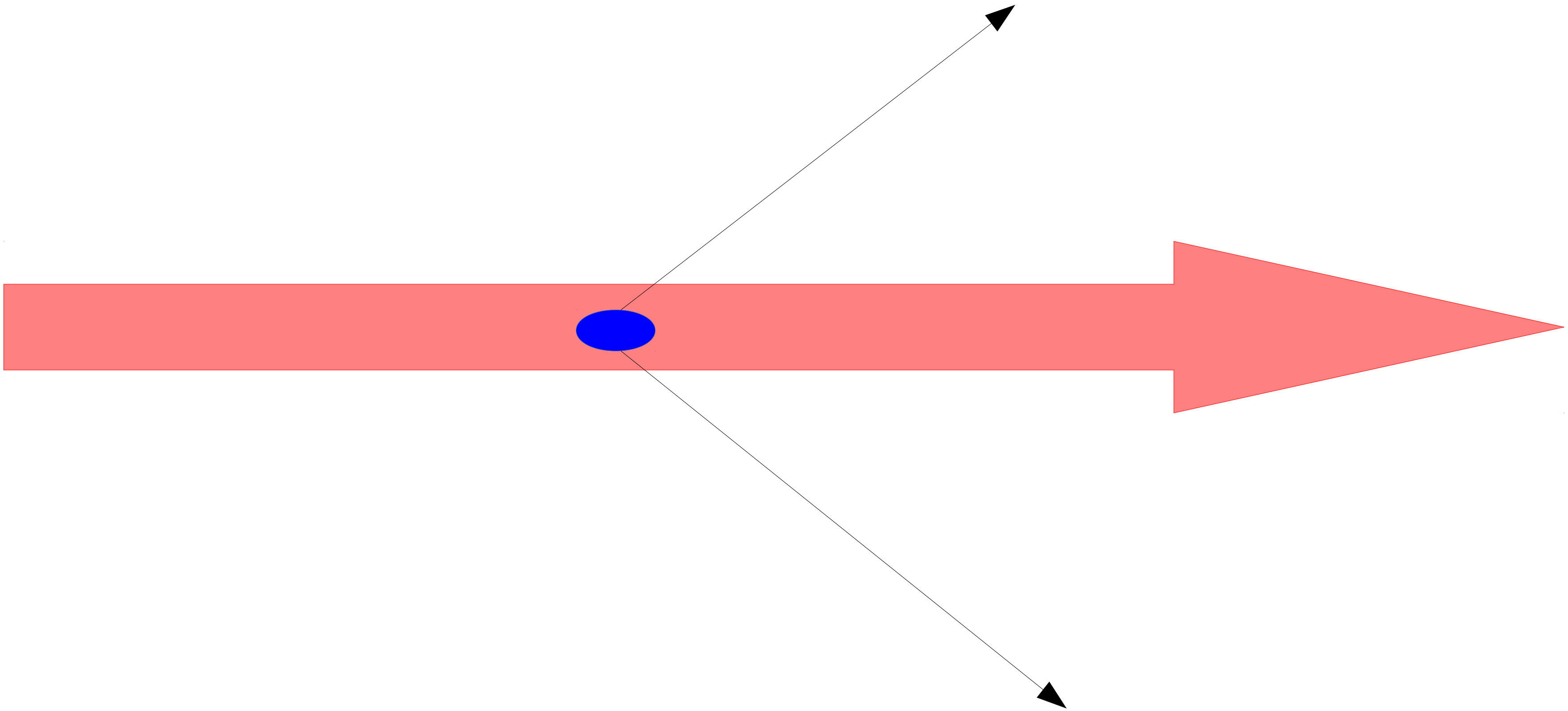
a) For wide-front illumination: try a non-circular (square) aperture in the far Fraunhofer zone

The important insight was made in M.I.Mishchenko, M.J. Berg, C.M. Sorensen and C.V.M. van der Mee, J.Quant.Spect.Rad.Trans. 110, 323, 2009

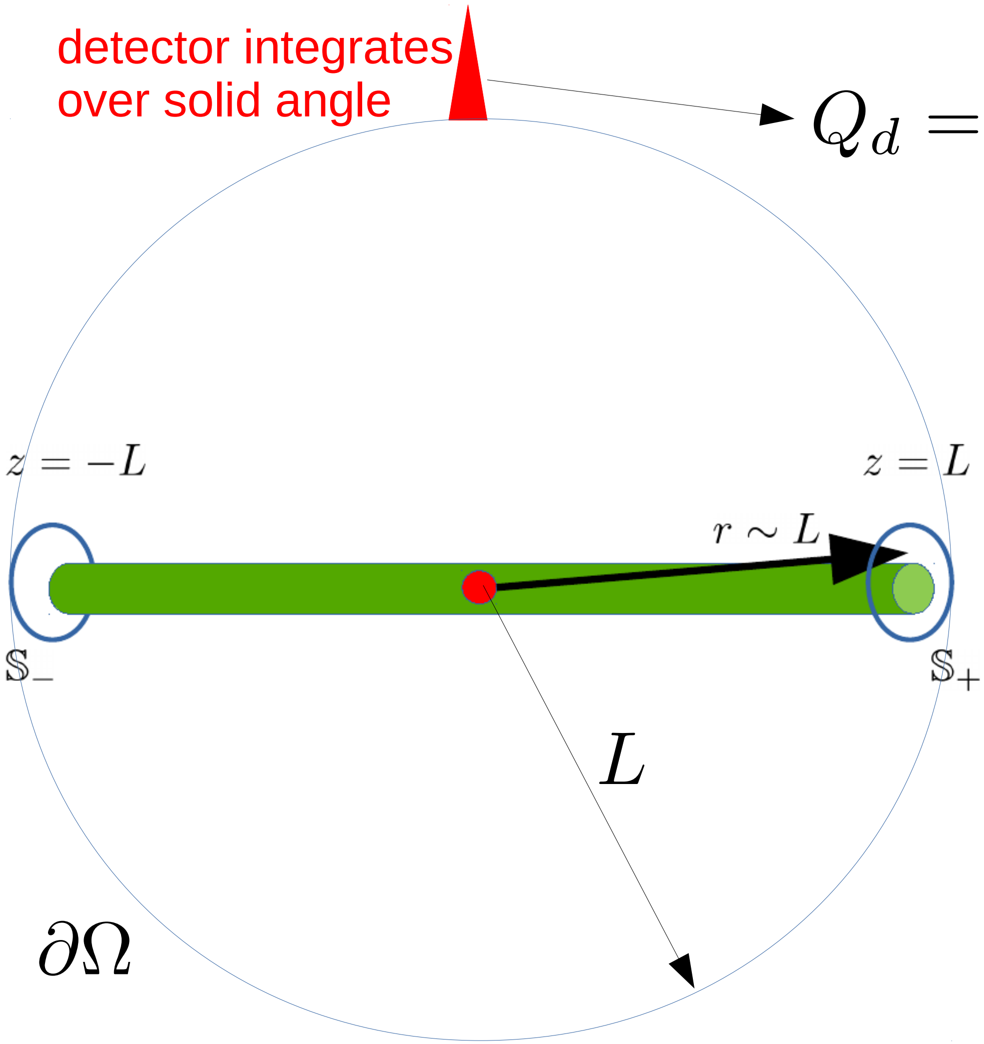




b) Let us try to separate scattered and incident fields by using a collimated beam



Paradox of extinction for collimated beams



$$Q_d = \oint_{\partial\Omega \setminus (\mathbb{S}_+ \cup \mathbb{S}_-)} [\mathbf{S}_t \cdot \hat{\mathbf{n}}] d^2 r$$

$$\xrightarrow[L\to\infty]{} \oint_{\partial\Omega} [\mathbf{S}_s \cdot \hat{\mathbf{n}}] d^2 r = Q_s$$

$$Q_e = \oint_{\mathbb{S}_-} [\mathbf{S}_{\rm int} \cdot \hat{\mathbf{z}}] d^2 r - \oint_{\mathbb{S}_+} [\mathbf{S}_{\rm int} \cdot \hat{\mathbf{z}}] d^2 r$$

$$L \to \infty \ : \ E_i, H_i \propto \text{const}$$
$$E_s, H_s \propto 1/L$$

$\Rightarrow Q_a < 0$?

$$Q_e \propto S E_i H_s \propto S E_0^2 / L \to 0$$

A resolution?

A pencil beam is an idealization; the beam must diverge

This is actually not enough; we can make a Gaussian beam arbitrarily well collimated by adjusting its waist.

If correct, that would mean that the scattered (and consequently extinguished) powers are not measurable.

Consideration complicated by the complex structure of vector Gauss beams. But, we have the same paradox for scalar waves.

Optical frequencies (500nm), waist 1mm --- beam would propagate without noticeable divergence for ~10meters – more than enough to demonstrate the paradox.

We will resolve the paradox for scalar waves, wherein it is conceptually the same. For vectorial EM waves the resolution is similar but technical details are more involved.

Scalar waves (wave function in QM scattering or acoustic scattering)

$$\Psi(\mathbf{r},t) = \psi(\mathbf{r})e^{-i\omega t}$$

$$\psi_t(\mathbf{r}) = \psi_i(\mathbf{r}) + \psi_s(\mathbf{r})$$

Wave function: $\Psi(\mathbf{r},t)$

Pressure field: $\mathrm{Re}[\Psi(\mathbf{r},t)]$

For monochromatic fields:

$$Q_a = -\oint_{\partial\Omega} [\mathbf{j}_t \cdot \hat{\mathbf{n}}] d^2r$$

$$\mathbf{j}_t(\mathbf{r}) = \mathrm{Im}[\psi_t^*(\mathbf{r})\nabla\psi_t(\mathbf{r})]$$

total current (of energy, probability, etc.)

$$Q_s = \oint_{\partial\Omega} [\mathbf{j}_s \cdot \hat{\mathbf{n}}] d^2r$$

$$\mathbf{j}_s(\mathbf{r}) = \mathrm{Im}[\psi_s^*(\mathbf{r})\nabla\psi_s(\mathbf{r})]$$

scattered current

$$Q_e = Q_a + Q_s = -\oint_{\partial\Omega} [\mathbf{j}_e \cdot \hat{\mathbf{n}}] d^2r$$

$$\mathbf{j}_e = \mathrm{Im}(\psi_i^*\nabla\psi_s + \psi_s^*\nabla\psi_i)$$

interference term in the definition of extinction

in the "interaction region"

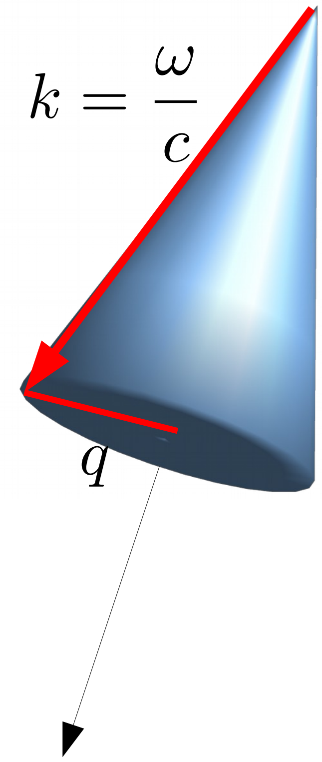
$$\psi_s(\mathbf{r}) = \alpha\psi_i(0)\frac{e^{ikr}}{r} \propto 1/L$$

Paradox is still present

Scalar Gaussian beam

$$\psi_i(\boldsymbol{\rho}, z) = \frac{1}{\pi k^2 \sigma^2} \int_0^\infty q dq \int_0^{2\pi} d\varphi_{\mathbf{q}}\; e^{-(q/\sigma k)^2} e^{iz\sqrt{k^2-q^2}} e^{i\mathbf{q}\cdot\boldsymbol{\rho}}$$

$$k = \frac{\omega}{c}$$



$$= \frac{2}{\pi\sigma^2} \int_0^\infty \xi d\xi\; e^{-(\xi/\sigma)^2} e^{i(kz)\sqrt{1-\xi^2}} J_0(k\rho\xi)$$

$$\psi_i(\mathbf{0}, 0) = 1$$

σ – scalar parameter (usually <<1)

Paraxial approximation:

$$kz \ll 4\pi/\sigma^4$$

$$\sqrt{1-\xi^2} \approx 1 - \frac{1}{2}\xi^2 - \frac{1}{8}\xi^4$$

$$\psi_i(\boldsymbol{\rho}, z) = \frac{1}{1 + i\sigma^2 kz/2} \exp\left[ikz - \frac{(\sigma k\rho/2)^2}{1 + i\sigma^2 kz/2}\right]$$

$$w(z) = w_0\sqrt{1 + \left(\frac{z}{z_0}\right)^2}, \quad w_0 = \frac{2}{\sigma k}, \quad z_0 = \frac{2}{\sigma^2 k}$$

Paraxial approximation (cont.)

So there exists a range of kz: $2/\sigma \ll kz \ll 2/\sigma^2$ in which:

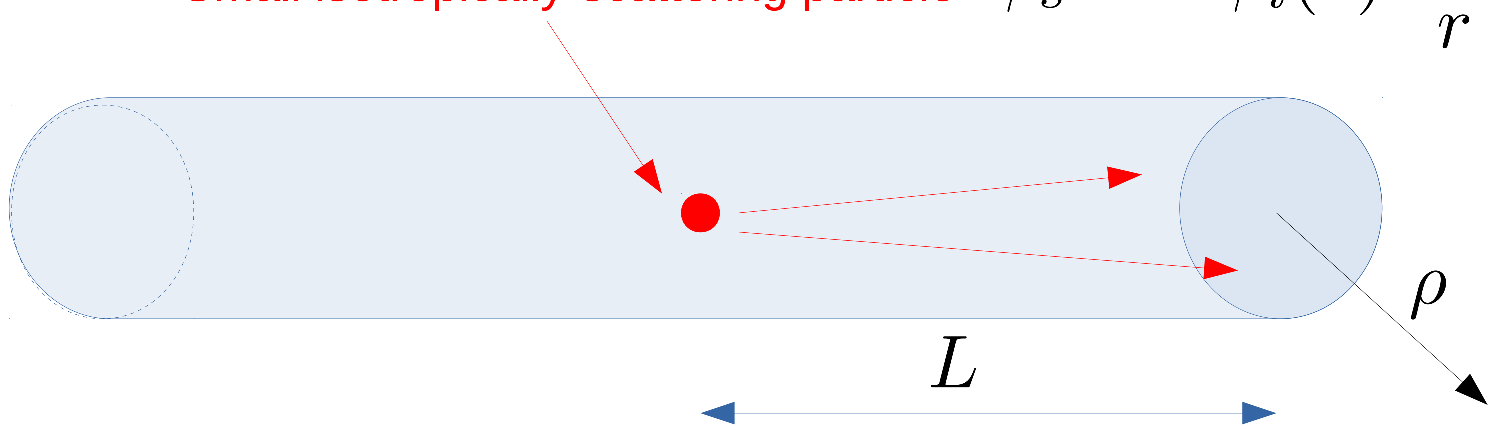
(i) $z \gg w_0$ (propagation distance $\gg$ waist size)

(ii) paraxial approximation is accurate

(iii) there is no noticeable diffraction (pencil beam is a good approx.)

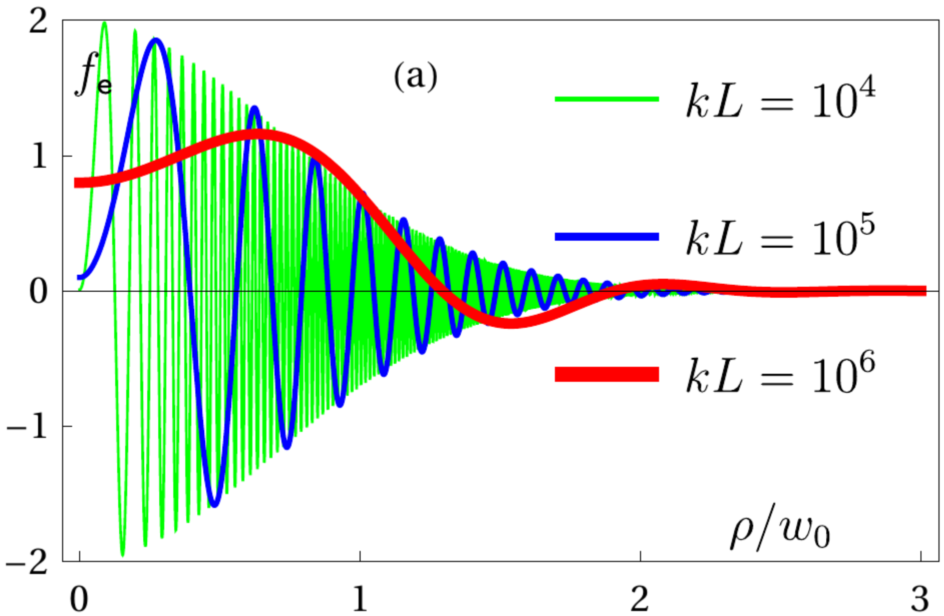
In simulations, we take $\sigma = 10^{-3}$ and $kL = 10^4,\ 10^5,\ 10^6$

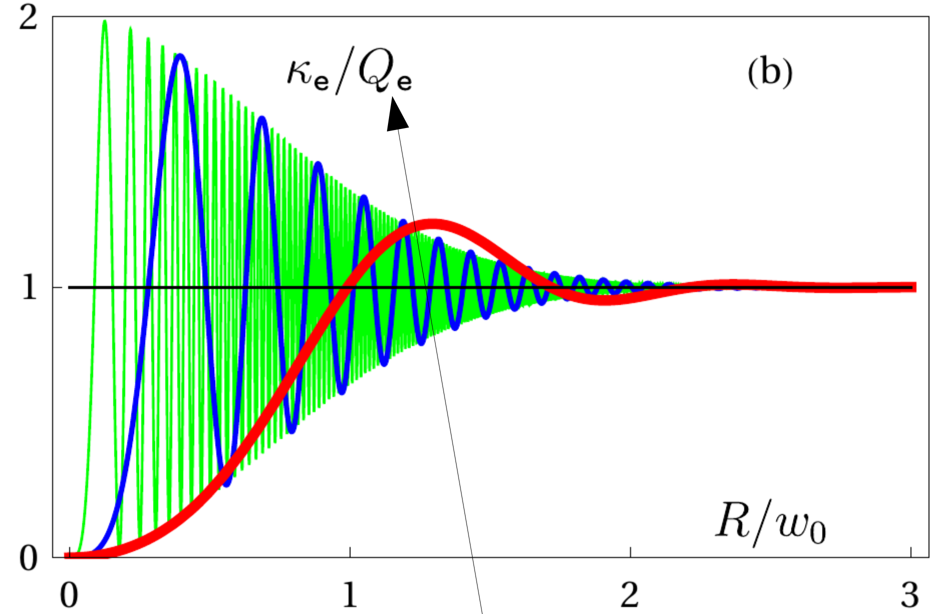
Small isotropically-scattering particle $\psi_s = \alpha \psi_i(0) \dfrac{e^{ikr}}{r}$

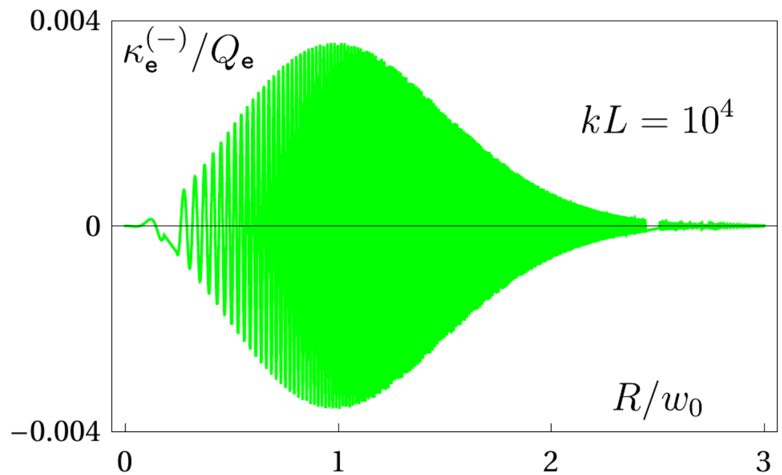


$$\kappa_e(R) = \int_{\mathbb{S}_- \cup \mathbb{S}_+} \mathbf{j}_e \cdot \hat{\mathbf{n}} d^2 r = \frac{2\pi}{L} \int_0^R f_e(\rho) \rho d\rho \ , \quad Q_e = \lim_{R \to \infty} \kappa_e(R)$$

Numerical evaluation of integrals







interference energy flux through the back-face z=-L

$$Q_e = 4\pi \mathrm{Im}\alpha$$

theoretical extinguished power (i.e., from optical theorem)

So, the "interaction area" stays approximately the same (a circle or radius w0) but oscillations of the integrand become slower, which exactly cancels the 1/*L* factor.

What about larger distances?

- We have considered the range $2/\sigma \ll kz \ll 2/\sigma^2$
- The paraxial approximation is valid for $kz \ll 4\pi/\sigma^4$
- The effect when the integrand becomes less oscillatory with L can not continue forever.
- So, when $2/\sigma^2 \ll kz \ll 4\pi/\sigma^4$, the interaction spot starts to increase as L
- For even larger distances, the paraxial approximation breaks down. Computing the highly oscillatory integrals becomes difficult or impossible – but conservation of energy still works

Now we have complete resolution of the paradox:

- In the Fresnel diffraction region, the beam divergence is small or negligible, and the $1/L$ dependence of the scattered field is canceled by the oscillatory nature of the integral
- At larger distances (in the Fraunhoffer zone) the area of the interaction region starts to increase
- At even larger distances, the paraxial approximation breaks down and interference occurs

Conclusions

- Extinction is a very robust property of particles to remove power from an incident beam or wide-front radiation
- However, some physical situations involving extinction are surprisingly complex
- All paradoxes can be resolved by working from first principles
- In the case of multiple scattering, extinction cross section of a particle is not sufficient forn a complete complete characterization of the medium.