Fourier Inversion of the Star Transform

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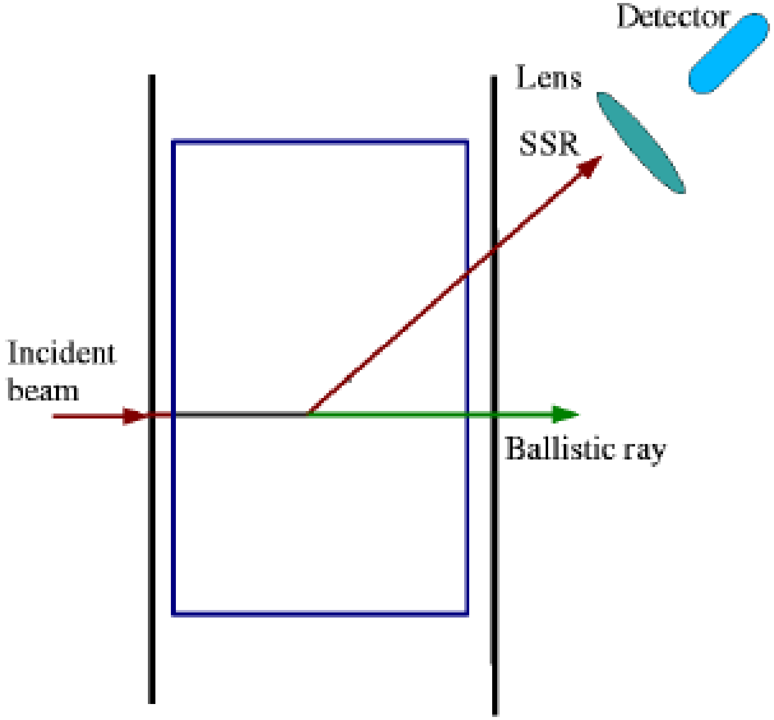
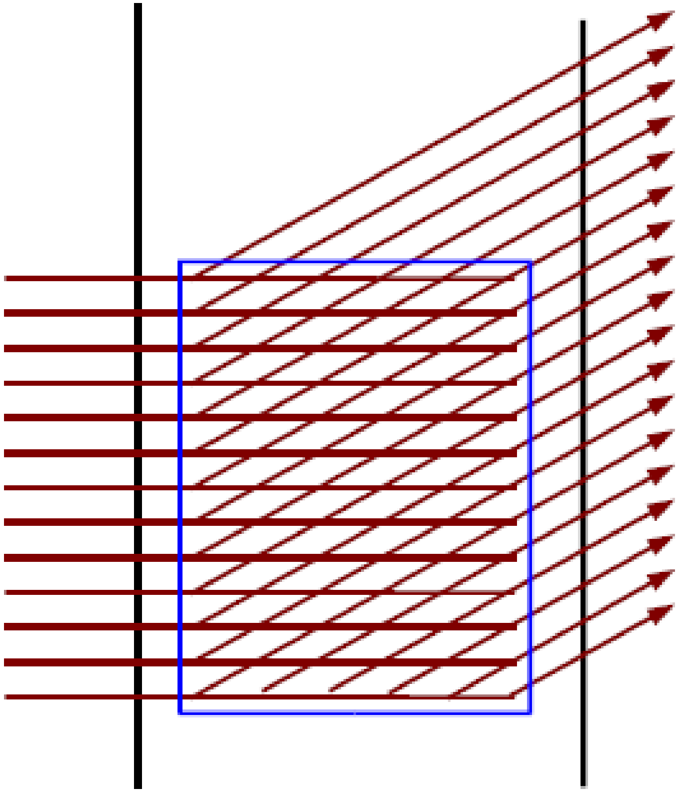
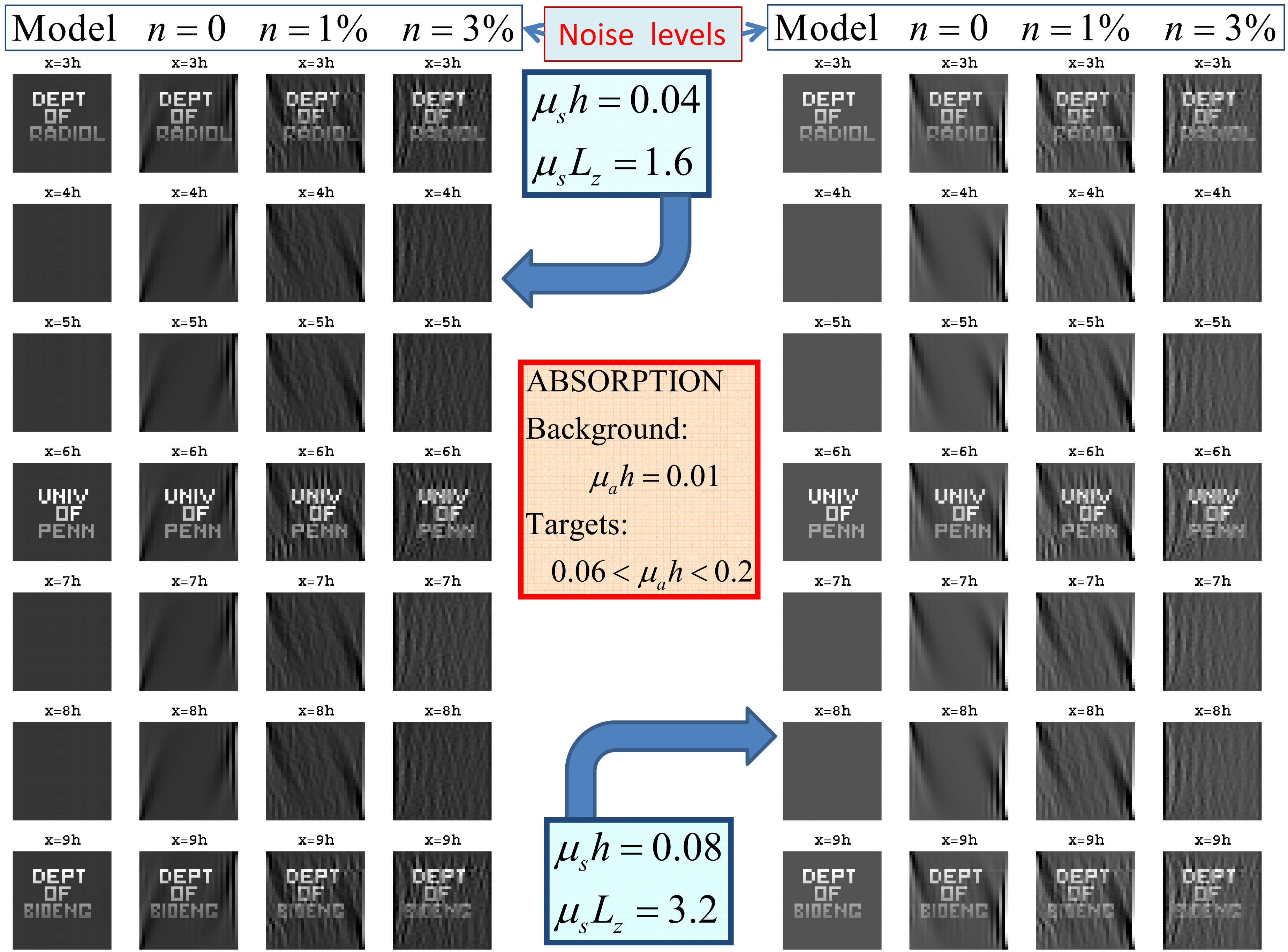
OUTLINE

1. Motivation







Model $n=0$ $n=1\%$ $n=3\%$
Noise levels
Model $n=0$ $n=1\%$ $n=3\%$
$\mu_s h = 0.04$
$\mu_s L_z = 1.6$
ABSORPTION
Background:
$\mu_a h = 0.01$
Targets:
$0.06 < \mu_a h < 0.2$
$\mu_s h = 0.08$
$\mu_s L_z = 3.2$
x=3h
x=4h
x=5h
x=6h
x=7h
x=8h
x=9h
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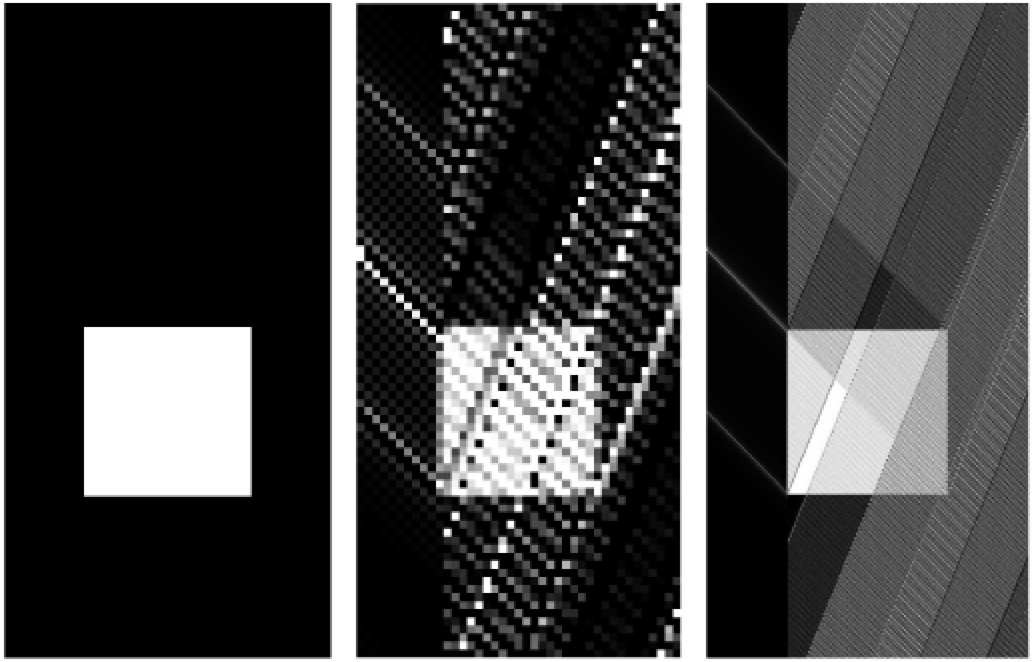
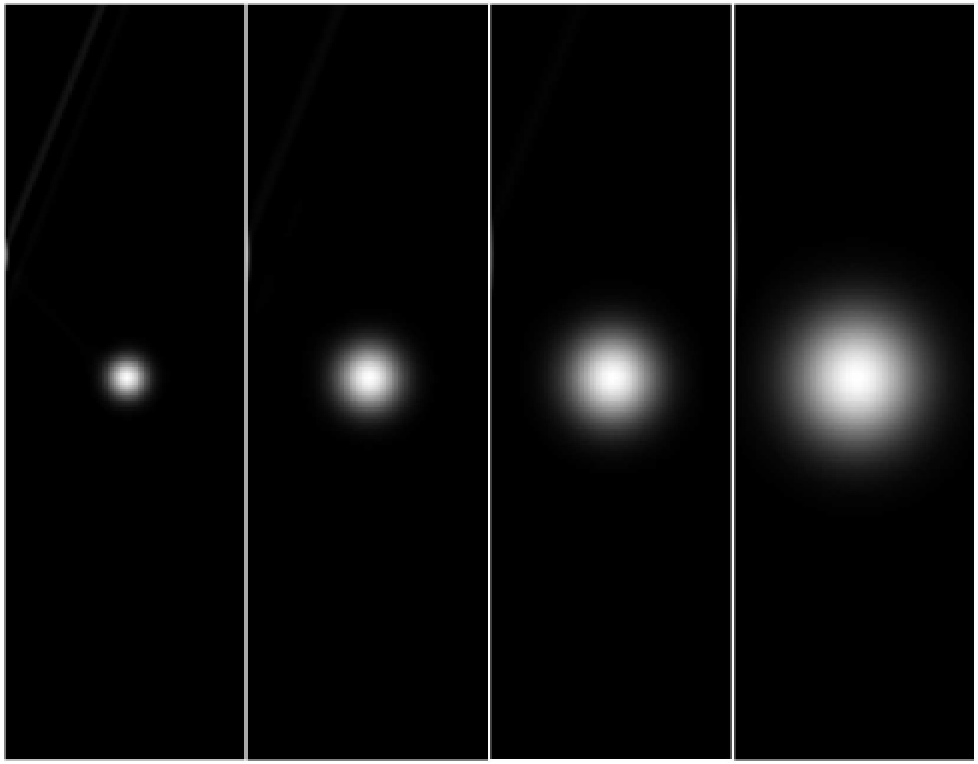
Analytical reconstruction formula for one scattering angle (inverse-crime simulations)

$$\beta = \frac{\pi}{4}$$



Model   *L/h*=40   *L/h*=400

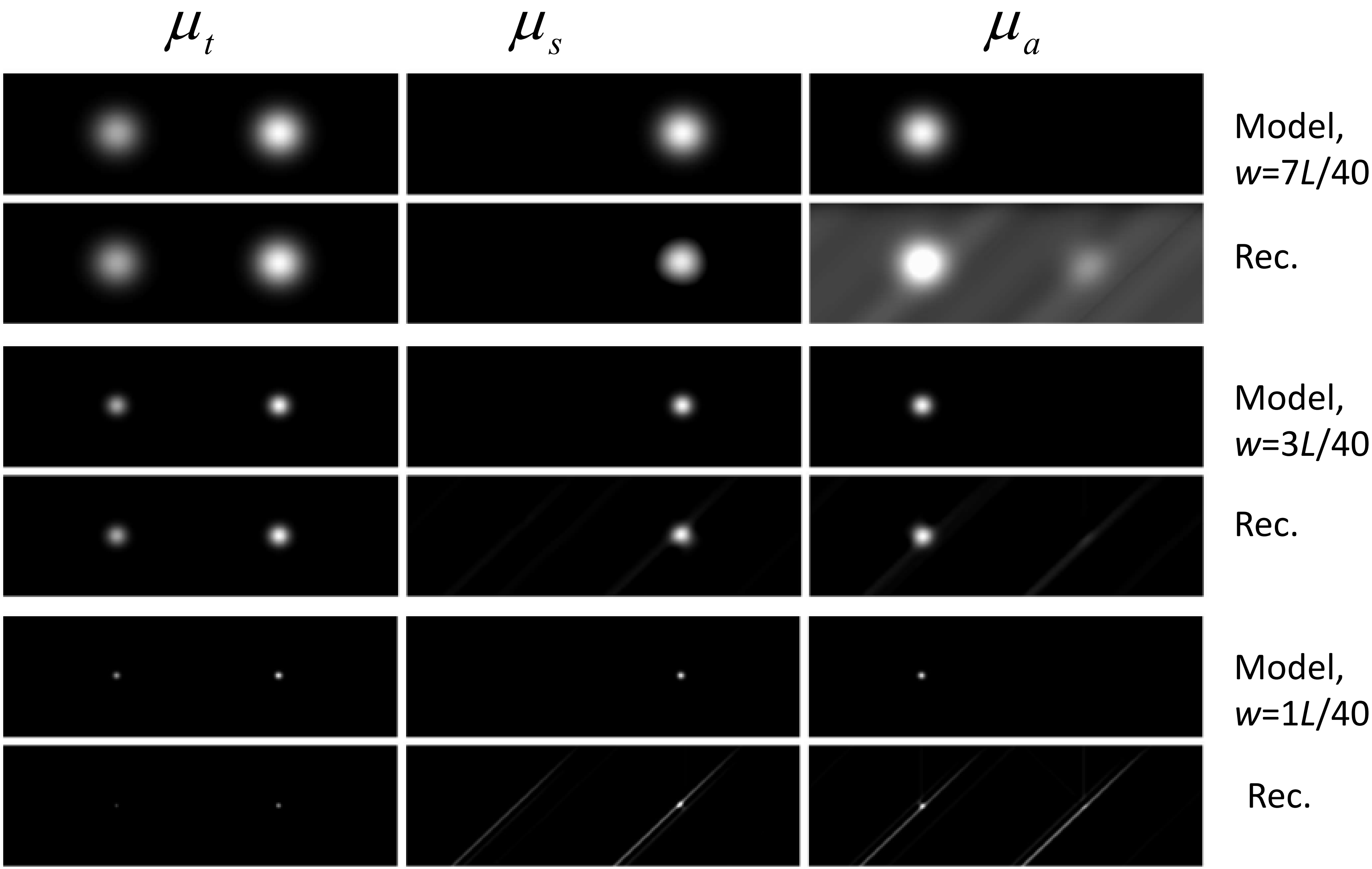
Reconstruction of a Gaussian



$$\frac{w}{L} = \frac{3}{40} \quad \frac{w}{L} = \frac{5}{40} \quad \frac{w}{L} = \frac{7}{40} \quad \frac{w}{L} = \frac{10}{40}$$

$$\beta = \frac{\pi}{4}$$

$$\frac{h}{L} = \frac{0.3}{40}$$



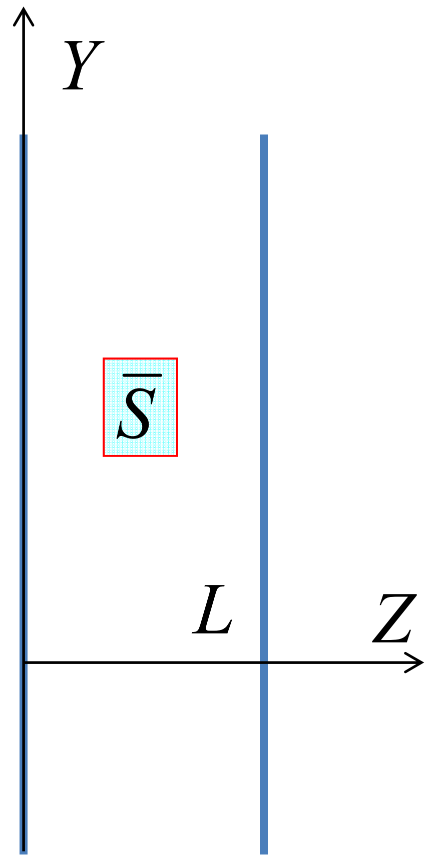
$$\frac{\bar{\mu}_s}{\bar{\mu}_a} = \frac{4}{3} \; ; \; \delta\mu_{a,s} = \bar{\mu}_{a,s} \exp\left[-\frac{(\boldsymbol{r} - \boldsymbol{r}_{a,s})^2}{w^2} \right] \; ; \quad \frac{h}{L} = \frac{0.3}{40}$$

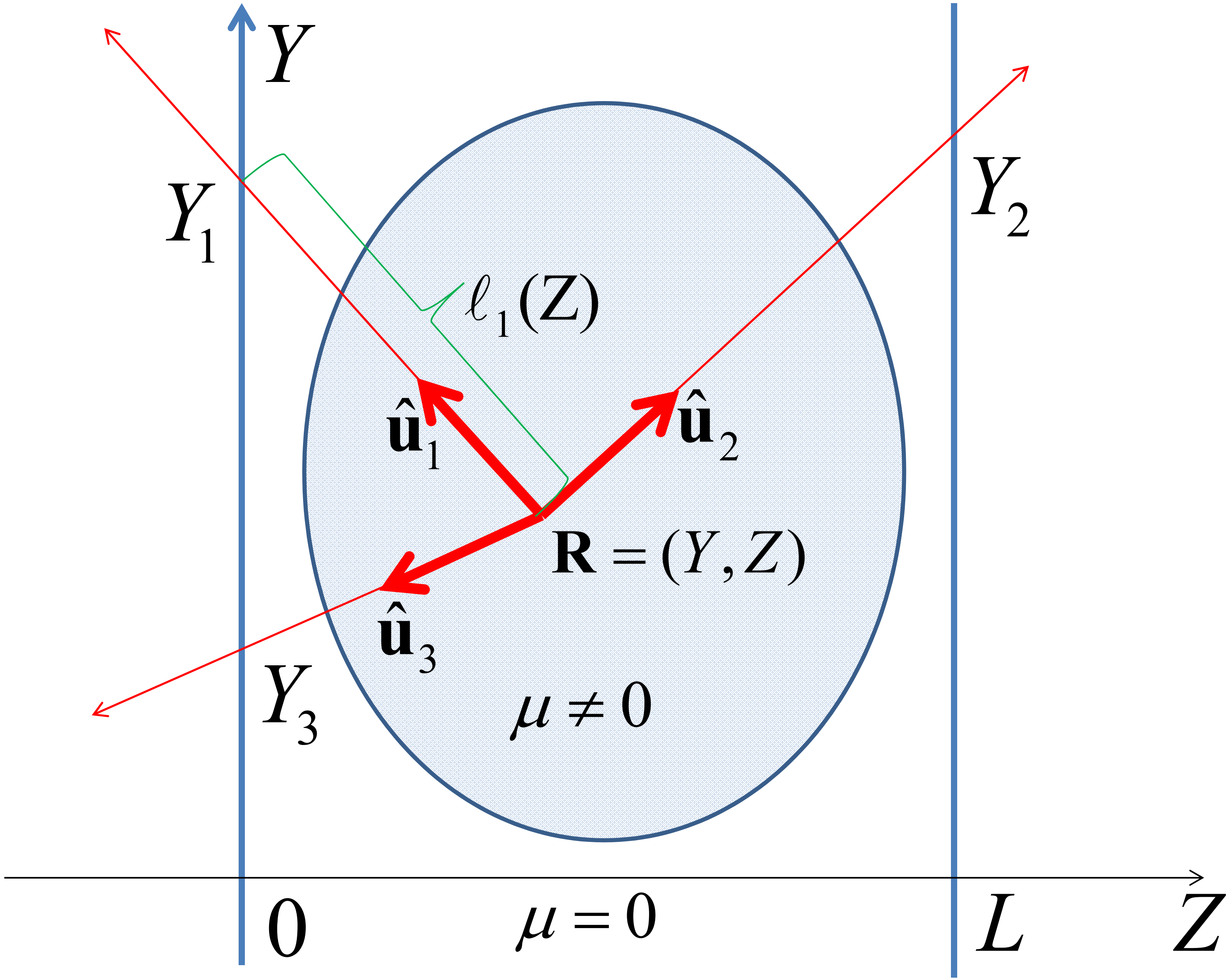
2. Definition of the Star Transform

$$\Phi(\mathbf{R}) = \sum_{k=1}^{K} s_k I_k(\mathbf{R}) \quad ,$$

$$\mathbf{R} \equiv (Y, Z) \in \bar{S} = \{0 \le Z \le L\} \quad ,$$

$$I_k(\mathbf{R}) = \int_0^{\ell_k(Z)} \mu(\mathbf{R} + \hat{\mathbf{u}}_k \ell) d\ell$$





Y
Y_1
Y_2
Y_3
$\ell_1(Z)$
$\hat{\mathbf{u}}_1$
$\hat{\mathbf{u}}_2$
$\hat{\mathbf{u}}_3$
$\mathbf{R} = (Y, Z)$
$\mu \neq 0$
0
$\mu = 0$
L
Z

3. Physical Derivation of Star Transform

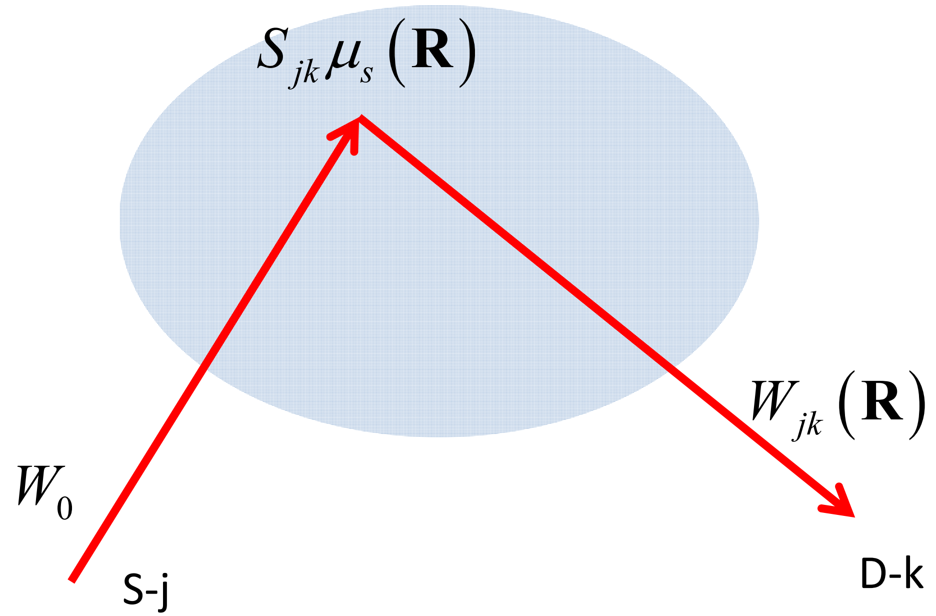
$$W_{jk}(\mathbf{R}) = W_0 S_{jk} \mu_s(\mathbf{R}) \exp\left[-I_j(\mathbf{R}) - I_k(\mathbf{R})\right]$$

$$\phi_{jk}(\mathbf{R}) = -\ln\left[\frac{W_{jk}(\mathbf{R})}{W_0 S_{jk} \bar{\mu}_s}\right]$$

$$\phi_{jk}(\mathbf{R}) = I_j(\mathbf{R}) + I_k(\mathbf{R}) + \eta(\mathbf{R})$$

where

$$\eta(\mathbf{R}) = -\ln\left(\frac{\mu_s(\mathbf{R})}{\bar{\mu}_s}\right)$$



$$\phi_{jk}(\mathbf{R}) = \left[I_j(\mathbf{R}) + I_k(\mathbf{R}) + \eta(\mathbf{R}) \right] \left(1 - \delta_{jk}\right) \qquad (1)$$

$j, k = 1, 2, \ldots, K$

$I_j(\mathbf{R})$ depend on total attenuation $\mu(\mathbf{R})$

$\eta(\mathbf{R})$ depends on the scattering coefficient $\mu_s(\mathbf{R})$

Strategy:

a) Exclude $\eta(\mathbf{R})$ from the equations by considering linear combitantions of $\phi_{jk}(\mathbf{R})$:

$$\Phi(\mathbf{R}) = \frac{1}{2} \sum_{j,\mathrm{k}=1}^{K} c_{jk} \phi_{jk}(\mathbf{R}) = \sum_{k=1}^{K} s_k I_k(\mathbf{R})$$

b) Solve for total attenuation.

c) Using the above result, compute the ray integrals $I_k(\mathbf{R})$.

d) Use any of the equations in (1) to compute $\eta(\mathbf{R})$.

$$\Phi(\mathbf{R}) = \frac{1}{2}\sum_{j,k=1}^{K} c_{jk}\phi_{jk}(\mathbf{R})$$

Conditions on c_{jk}

$(i)\ c_{jk} = c_{kj}$

$(ii)\ c_{kk} = 0$

$(iii)\ \sum_{j,k=1}^{K} c_{jk} = 0$

$(iv)\ \sum_{j=1}^{K} c_{jk} = s_k \neq 0$

$$\Phi(\mathbf{R}) = \sum_{k=1}^{K} s_k I_k(\mathbf{R})$$

0	1	1	2
1	0	−2	−1
1	−2	0	−1
2	−1	−1	0

0	1	1	−1	1
1	0	−1	−1	−1
1	−1	0	1	1
−1	−1	1	0	−1
1	−1	1	−1	0

This last conditions is not critical.
It excludes the possibility of star transforms in which a ray integral has zero "weight"

4. Local Methods

$$-\left(\hat{\mathbf{u}}_k \cdot \nabla\right) I_k\left(\mathbf{R}\right) = \mu\left(\mathbf{R}\right)$$

Unfortunately, we can not make measurements of ray integrals $I_k\left(\mathbf{R}\right)$ directly. However, we can formulate the star transfrom so that the coefficients $\mathbf{s}_k$ and $\mathbf{\Phi}$ are vectors. Then it is possible to invert the star transform by the local formula

$$\mu\left(\mathbf{R}\right) = \nabla \cdot \mathbf{\Phi}\left(\mathbf{R}\right)$$

[Katsevich and Krylov, Inverse Problems **29**, 075008 (2013)]

What if we allow the coefficients $\mathbf{c}_{jk}$ to be vectors?

Moreover, let

$$\sum_{j=1}^{K} \mathbf{c}_{jk} = \mathbf{s}_k = \sigma_k \hat{\mathbf{u}}_k \quad \text{and} \quad \sum_{k=1}^{K} \mathbf{s}_k = \sum_{k=1}^{K} \sigma_k \hat{\mathbf{u}}_k = 0$$

Then define

$$\boldsymbol{\Phi}(\mathbf{R}) \equiv \frac{1}{2} \sum_{j,k=1}^{K} \mathbf{c}_{jk} \phi_{jk}(\mathbf{R})$$

and

$$\boldsymbol{\Phi}(\mathbf{R}) = \sum_{k=1}^{K} \sigma_k \hat{\mathbf{u}}_k I_k(\mathbf{R})$$

$$-\nabla \cdot \boldsymbol{\Phi}(\mathbf{R}) = \left(\sum_{k=1}^{K} \sigma_k \right) \mu(\mathbf{R})$$

$$\mu(\mathbf{R}) = -\frac{1}{\sum_{k=1}^{K} \sigma_k} \nabla \cdot \boldsymbol{\Phi}(\mathbf{R})$$

$\mathbf{c}_{jk}$ matrix for $K = 3$

$$
\left[\begin{array}{ccc|c}
0 & \sigma_1\hat{\mathbf{u}}_1 + \sigma_2\hat{\mathbf{u}}_2 & \sigma_1\hat{\mathbf{u}}_1 + \sigma_3\hat{\mathbf{u}}_3 & \sigma_1\hat{\mathbf{u}}_1 \\
\sigma_1\hat{\mathbf{u}}_1 + \sigma_2\hat{\mathbf{u}}_2 & 0 & \sigma_2\hat{\mathbf{u}}_2 + \sigma_3\hat{\mathbf{u}}_3 & \sigma_2\hat{\mathbf{u}}_2 \\
\sigma_1\hat{\mathbf{u}}_1 + \sigma_3\hat{\mathbf{u}}_3 & \sigma_2\hat{\mathbf{u}}_2 + \sigma_3\hat{\mathbf{u}}_3 & 0 & \sigma_3\hat{\mathbf{u}}_3 \\
\hline
\sigma_1\hat{\mathbf{u}}_1 & \sigma_2\hat{\mathbf{u}}_2 & \sigma_3\hat{\mathbf{u}}_3 & 0
\end{array}\right]
$$

Here σ_k are chosen so that $\sigma_1\hat{\mathbf{u}}_1 + \sigma_2\hat{\mathbf{u}}_2 + \sigma_3\hat{\mathbf{u}}_3 = 0$

Reconstruction formula:

$$\mu = -\frac{1}{\sigma_1 + \sigma_2 + \sigma_3}\nabla \cdot \left[\sigma_1\hat{\mathbf{u}}_1\left(\phi_{12} + \phi_{13}\right) + \sigma_2\hat{\mathbf{u}}_2\left(\phi_{21} + \phi_{23}\right) + \sigma_3\hat{\mathbf{u}}_3\left(\phi_{31} + \phi_{32}\right)\right]$$

Symmetric case $\hat{\mathbf{u}}_1 + \hat{\mathbf{u}}_2 + \hat{\mathbf{u}}_3 = 0$:

$$\mu = -\frac{1}{3}\nabla \cdot \left[\hat{\mathbf{u}}_1\left(\phi_{32} - \phi_{21}\right) + \hat{\mathbf{u}}_2\left(\phi_{31} - \phi_{12}\right)\right]$$

c_{jk} matrix for $K = 4$ (method of Katsevich and Krylov)

$\sigma_1 = 0, \sigma_2 = 1, \sigma_3 = -a, \sigma_4 = -b$

$-a - b = 0$

$\hat{\mathbf{u}}_2 - a\hat{\mathbf{u}}_3 - b\hat{\mathbf{u}}_4 = 0$

$$
\left[\begin{array}{cccc|c}
0 & \hat{\mathbf{u}}_2 & -a\hat{\mathbf{u}}_3 & -b\hat{\mathbf{u}}_4 & 0 \\
\hat{\mathbf{u}}_2 & 0 & 0 & 0 & \hat{\mathbf{u}}_2 \\
-a\hat{\mathbf{u}}_3 & 0 & 0 & 0 & -a\hat{\mathbf{u}}_3 \\
-b\hat{\mathbf{u}}_4 & 0 & 0 & 0 & -b\hat{\mathbf{u}}_4 \\
\hline
0 & \hat{\mathbf{u}}_2 & -a\hat{\mathbf{u}}_3 & -b\hat{\mathbf{u}}_4 & 0
\end{array}\right]
$$

Reconstruction formula:

$$\mu = \frac{1}{a + b - 1} \nabla \cdot \left[\hat{\mathbf{u}}_2 \phi_{12} - a\hat{\mathbf{u}}_3 \phi_{13} - b\hat{\mathbf{u}}_4 \phi_{14} \right]$$

5. Fourier Methods

$$\Phi(\mathbf{R}) = \sum_{k=1}^{K} s_k I_k(\mathbf{R}) \quad , \quad I_k(\mathbf{R}) = \int_0^{\ell_k(Z)} \mu(\mathbf{R} + \hat{\mathbf{u}}_k \ell) d\ell$$

$$\mu(y,z) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iqy} \tilde{\mu}(q,z) = \int_{-\infty}^{\infty} \frac{dq}{2\pi} e^{iqy} \frac{1}{L} \sum_{n=-\infty}^{\infty} \mu_n(q) e^{i\kappa_n z} ,$$

$$\mu_n(q) = \int_{-\infty}^{\infty} dy e^{-iqy} \int_0^L dz e^{-i\kappa_n z} \mu(y,z) ,$$

$$\kappa_n = \frac{2\pi n}{L} .$$

Fix q. For each q, we obtain a set of linear equations (different q's are not mixed)

$$d_n \mu_n + \sum_{k=1}^{K} s_k \alpha_k \frac{1}{\beta_k + \kappa_n} \sum_{m=-\infty}^{\infty} \frac{\mu_m}{\beta_k + \kappa_m} = \Phi_n$$

$$\alpha_k = \frac{e^{i\beta_k \xi_k}\left(e^{-i\beta_k L} - 1\right)}{L u_{kz}}, \qquad \xi_k = 0, L$$

$$\beta_k = q \frac{u_{ky}}{u_{kz}}$$

$$d_n = \sum_{k=1}^{K} \frac{i s_k}{u_{kz}(\beta_k + \kappa_n)} = \sum_{k=1}^{K} \frac{i s_k}{\hat{\mathbf{u}}_k \cdot (q, \kappa_n)}$$

In matrix notations:

$$D\left|\mu\right\rangle+\sum_{k=1}^{K}s_k\alpha_k\left|\psi_k\right\rangle\left\langle\psi_k\right|\mu\rangle=\left|\Phi\right\rangle$$

D is diagonal

$\left|\psi_k\right\rangle$ are known vectors

6. Analysis of Stability

a) $qL \ll 1$: Can be easily analyzed

b) $qL \gg 1$: Main concern. Not so simple but can also be analyzed

c) $qL \sim 1$: No analytical condition obtained; empirically, we have found that, if all rays closs the same boundary (reflection geometry), there can exist an instability in reconstructions. However, corresponding artifacts are localized near the the boundaries and are not of major concern.

a) Case $q = 0$

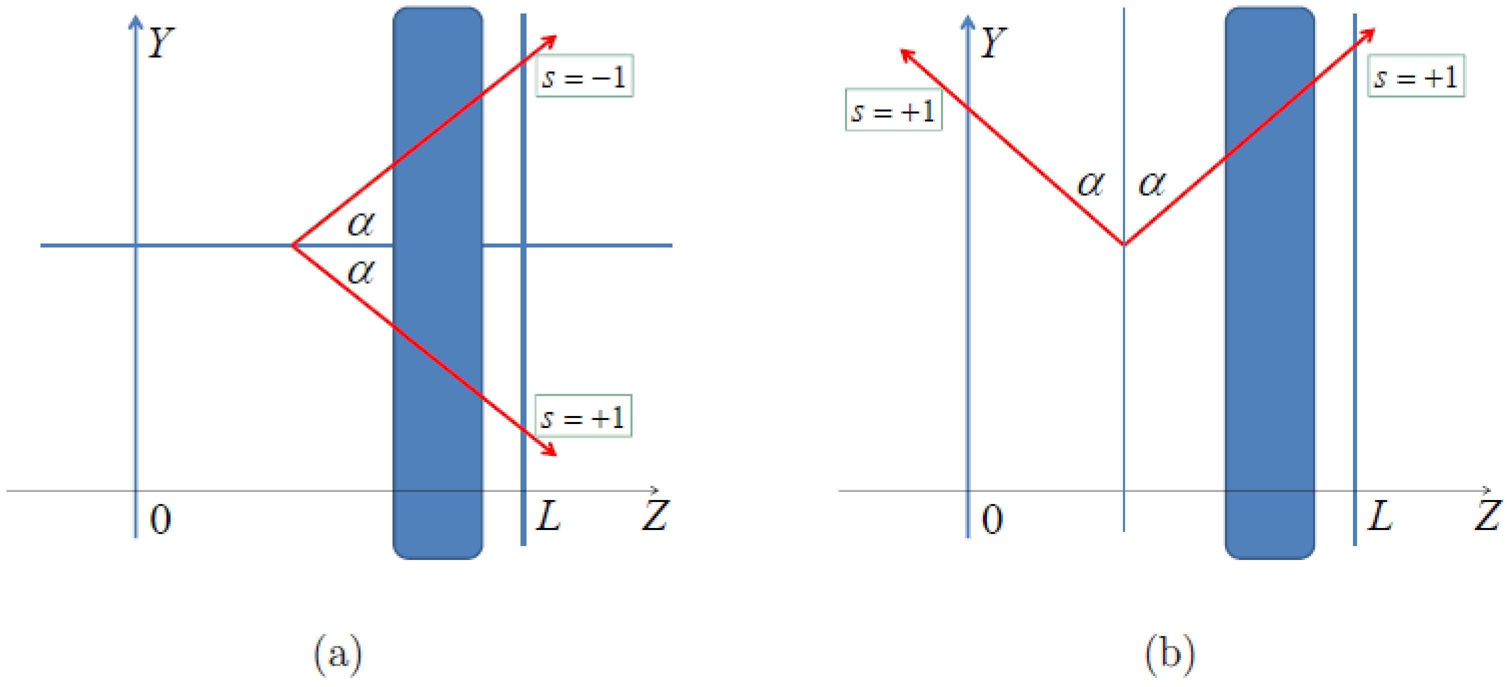
$$\mu_0 = \frac{2}{L\Sigma_0} \sum_{m=-\infty}^{\infty} \Phi_m = \frac{2\tilde{\Delta}(0)}{\Sigma_0} , \qquad n = 0 ,$$

$$\mu_n = \mu_0 - i\frac{\kappa_n \Phi_n}{\Sigma_1} , \qquad n \neq 0 .$$

$$\Sigma_0 = \sum_{k=1}^{K} \frac{s_k}{|u_{kz}|} , \quad \Sigma_1 = \sum_{k=1}^{K} \frac{s_k}{u_{kz}} , \quad \Sigma_2 = -\sum_{k=1}^{K} \frac{s_k u_{ky}}{u_{kz}^2} .$$

$$d_n = i\left(\Sigma_1 \frac{1}{\kappa_n} + \Sigma_2 \frac{q}{\kappa_n^2} + \ldots\right) , \quad |n| \to \infty ,$$

Bad imaging geometries (Sigma_1=0)



b) Case $q \to \infty$

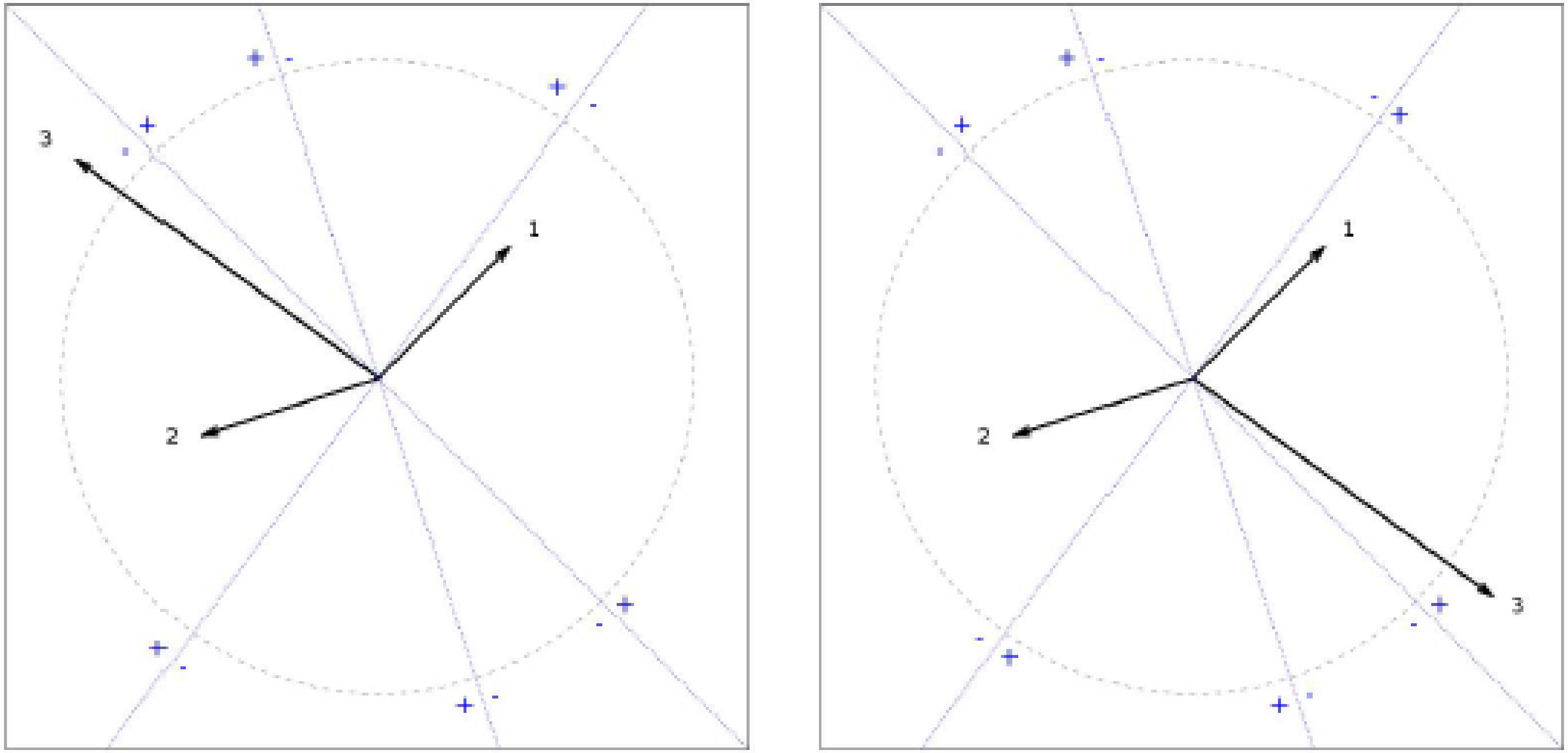
The diagonal matrix D somonates the separable terms

Find the condition under which all elements d_n of D are simultaneously non-zero

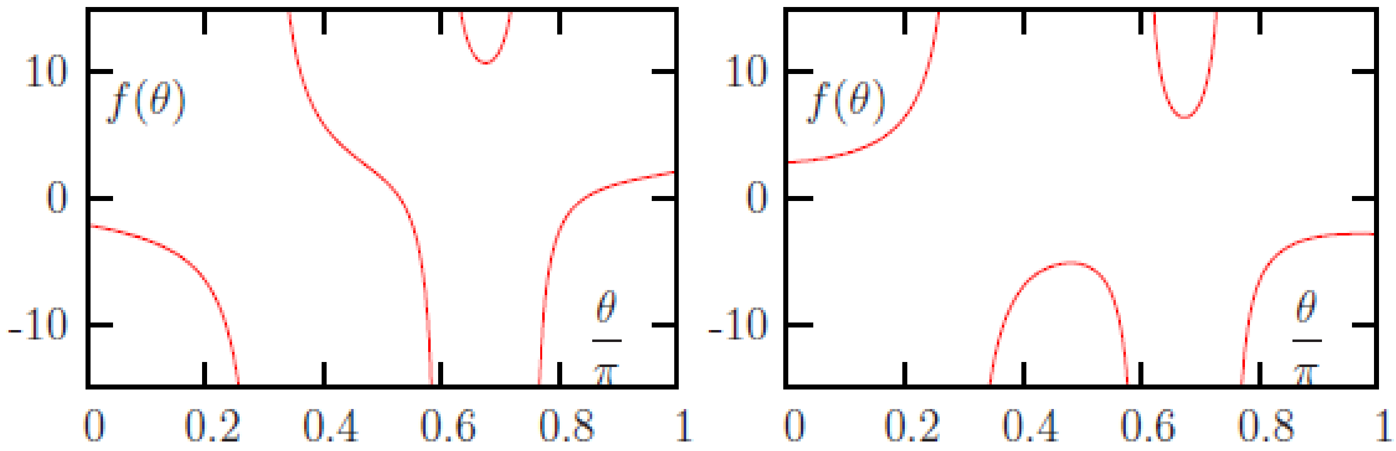
$$d_n(q) = \sum_{k=1}^{K} \frac{i s_k}{\hat{\mathbf{u}}_k \cdot (q, \kappa_n)} = \frac{i}{|(q, \kappa_n)|} \sum_{k=1}^{K} \frac{s_k}{\hat{\mathbf{u}}_k \cdot \hat{\mathbf{v}}}$$

$$f(\theta) = \sum_{k=1}^{K} \frac{s_k}{\cos(\theta - \theta_k)},$$

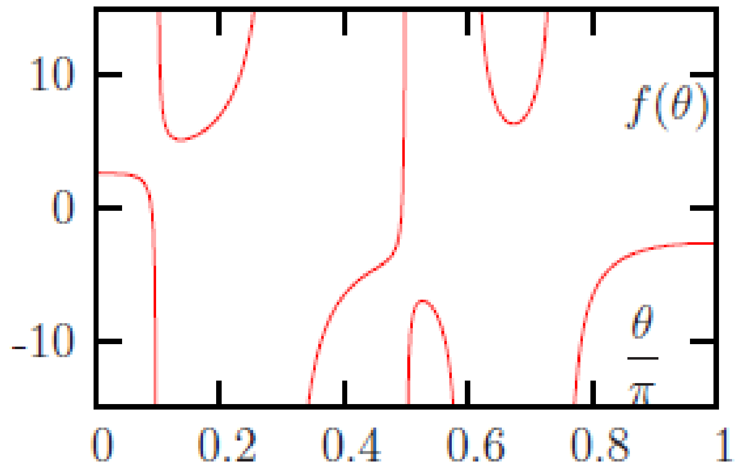
$f(\theta)$ has zeros $\Leftrightarrow$ at least one of the elements $d_n(q)$ turns to zero for some q and n. There exists a high-frequency instability



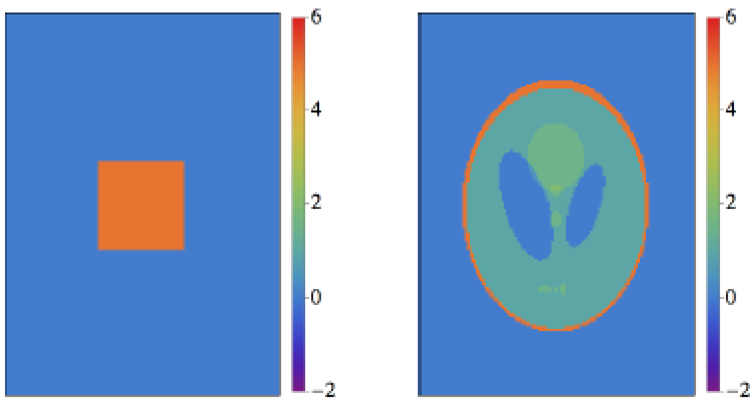
(a) (b)


10
0
-10
$f(\theta)$
$\frac{\theta}{\pi}$
0
0.2
0.4
0.6
0.8
1
10
0
-10
$f(\theta)$
$\frac{\theta}{\pi}$
0
0.2
0.4
0.6
0.8
1

(a) (b)


10
0
-10
$f(\theta)$
$\frac{\theta}{\pi}$
0
0.2
0.4
0.6
0.8
1

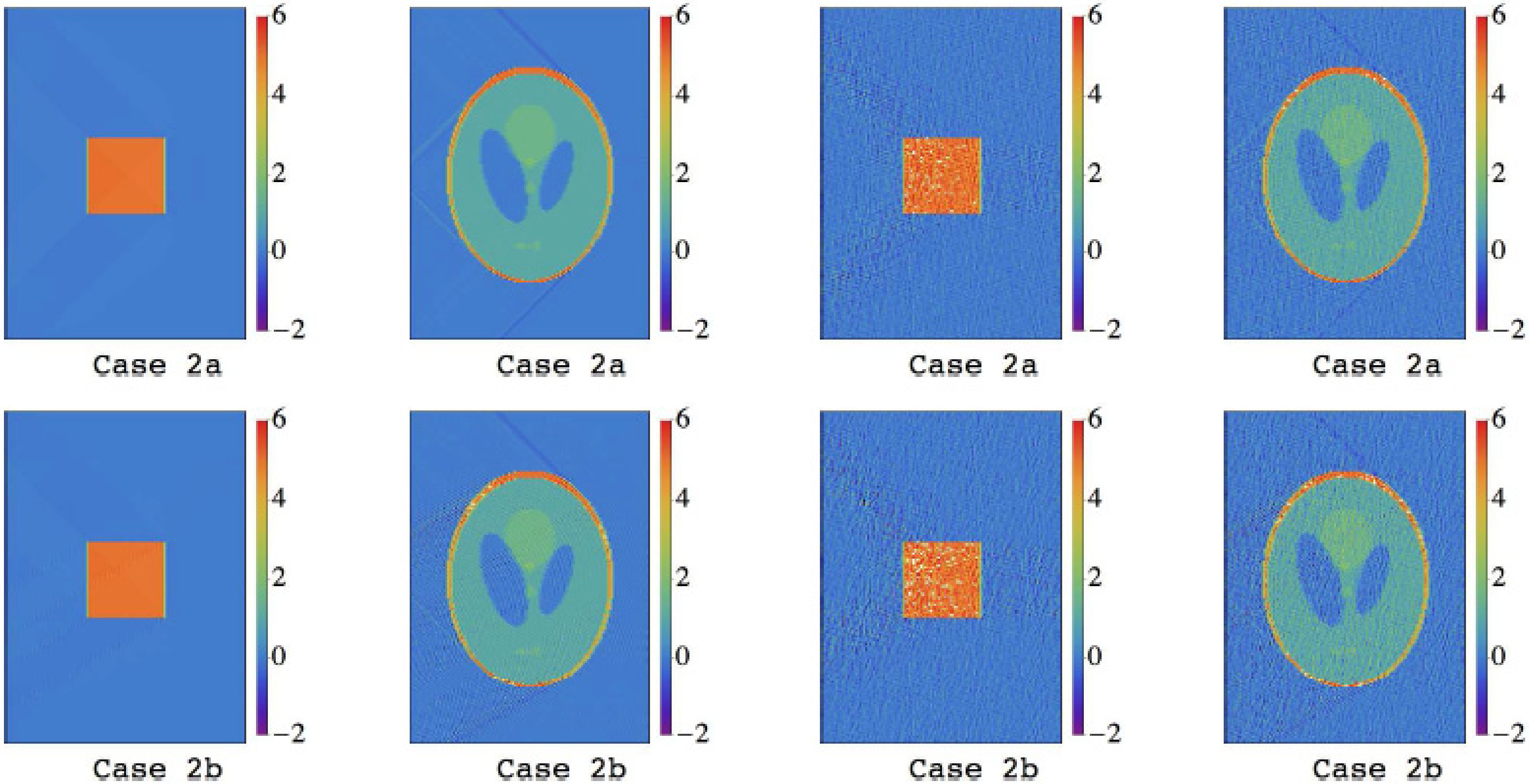
7. Numerical Examples



Phantoms used

Case	2a	2b	3a	3b
K	3		3	
s_1	1		1	
s_2	1		1	
s_3	1		-2	
θ_1/π	1		0.25	
θ_2/π	0.25		1.1	
θ_3/π	-0.25	-1/6	-0.2	0.8
NZ	0	0	2	0
Σ_0	3.83	3.57	-0.01	-0.01
Σ_1	1.83	1.57	-2.11	2.83

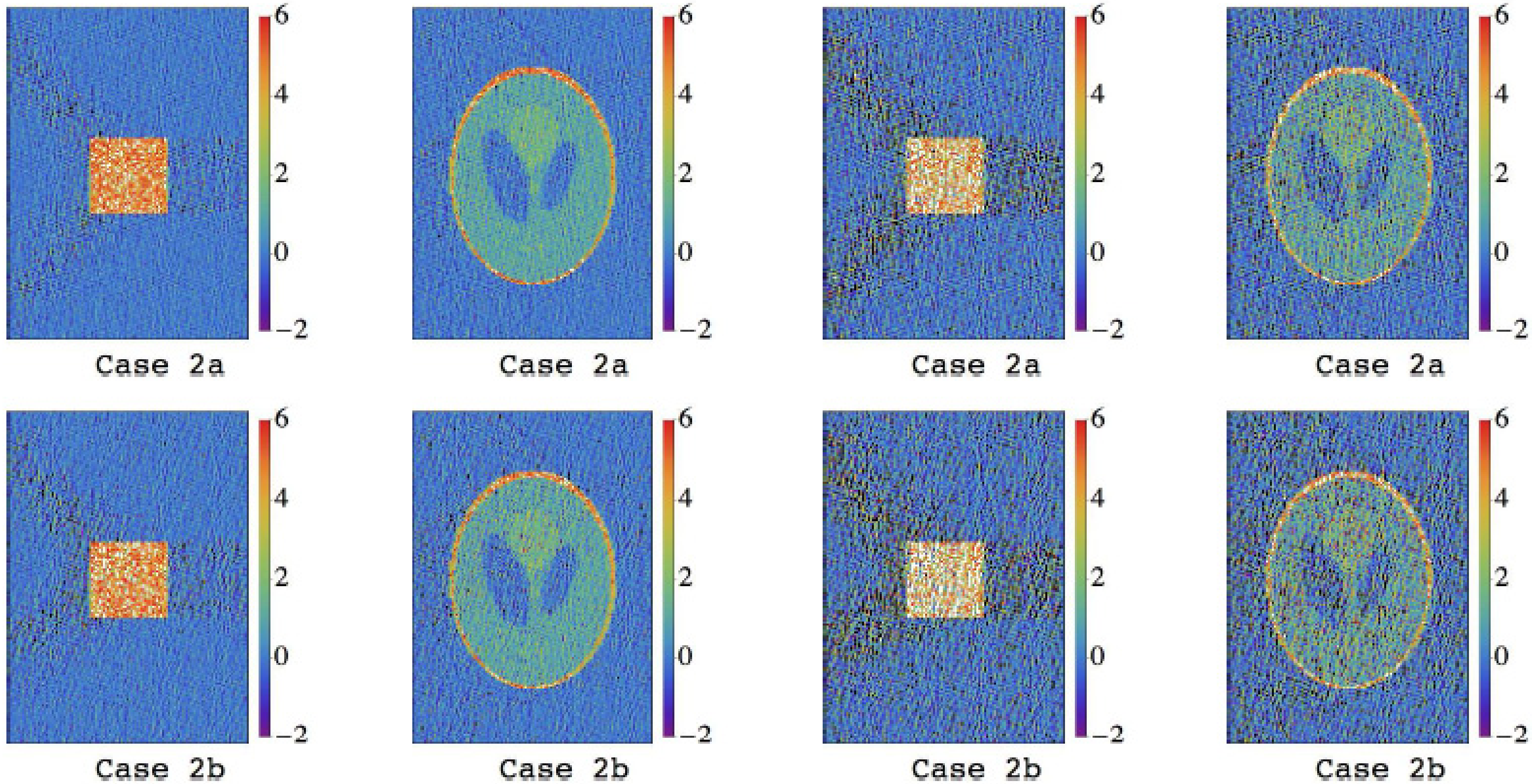
CASE 2; K=3, s1=s2=s3=1



(a) No noise, $\lambda = 0$

(b) $\mathcal{N} = 4 \times 10^4$, $\lambda = 0$

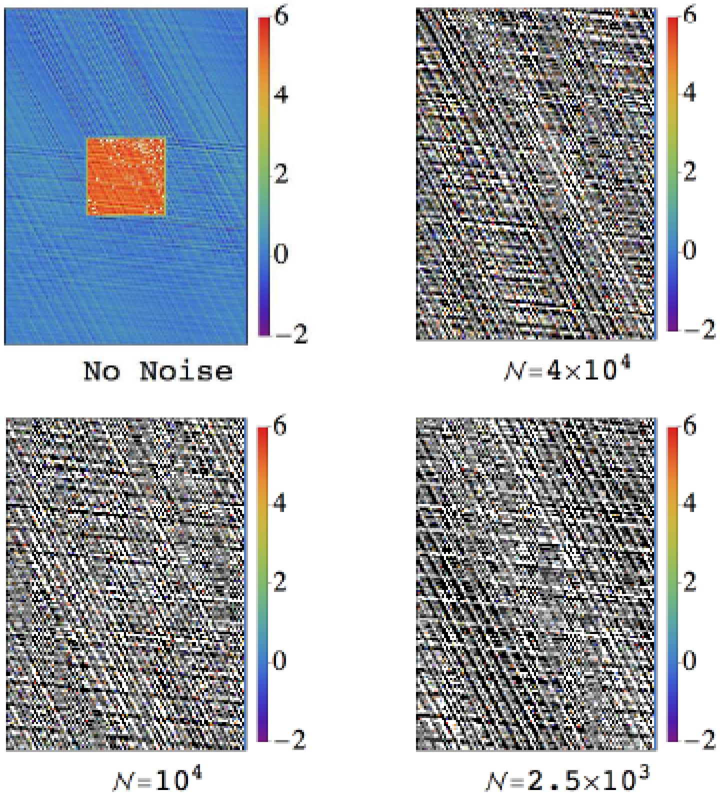
CASE 2; K=3, s1=s2=s3=1



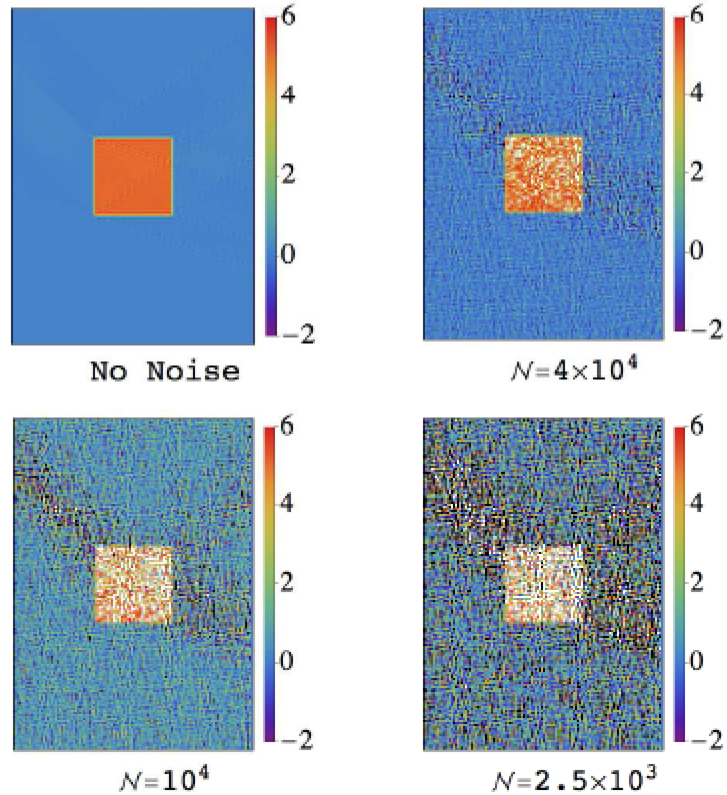
(c) $\mathcal{N} = 10^4$, $\lambda = 0$

(d) $\mathcal{N} = 2.5 \times 10^3$, $\lambda = 0$

CASE 3; K=3, s1=s2=1, s3=-2

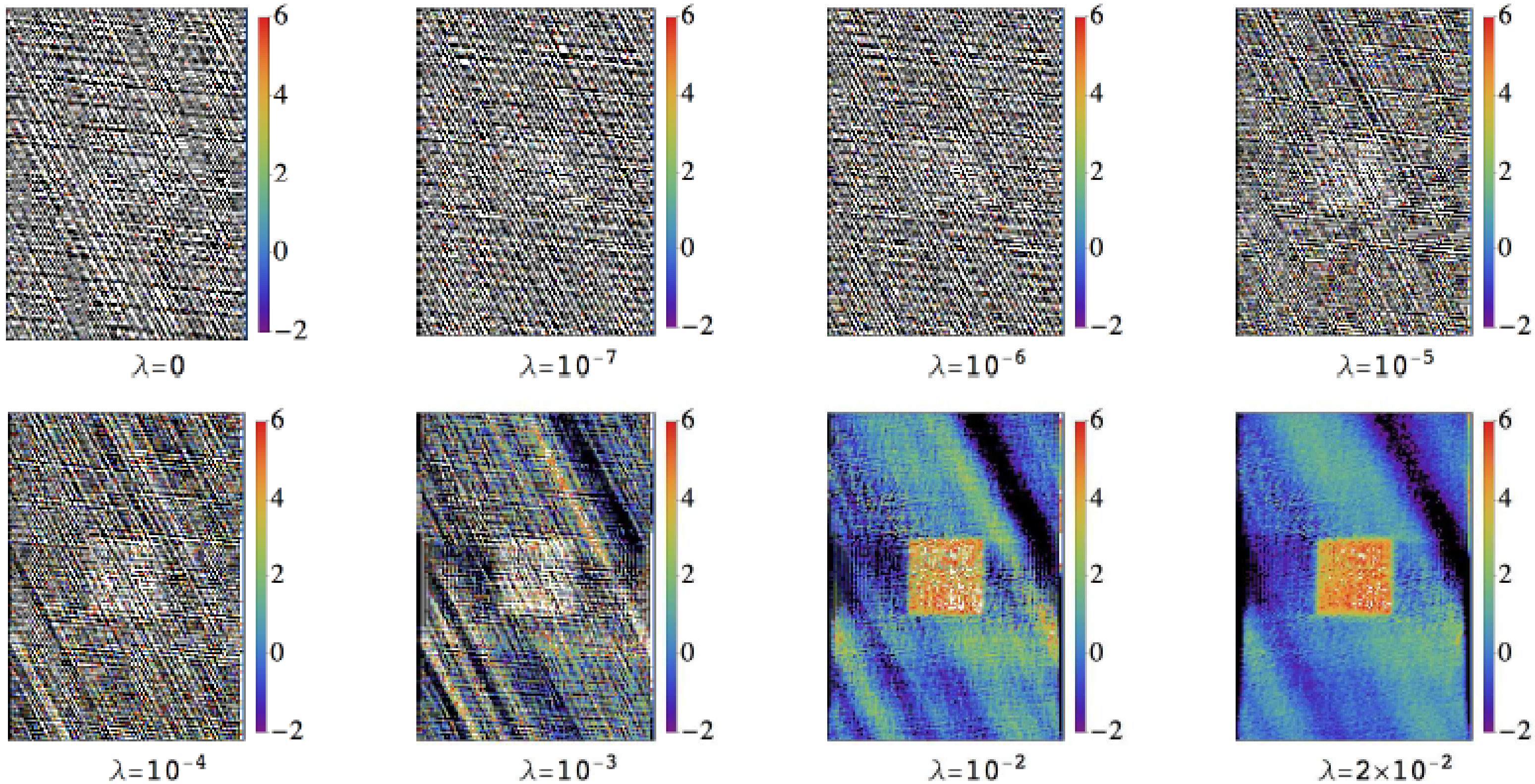


(a) Case 3a



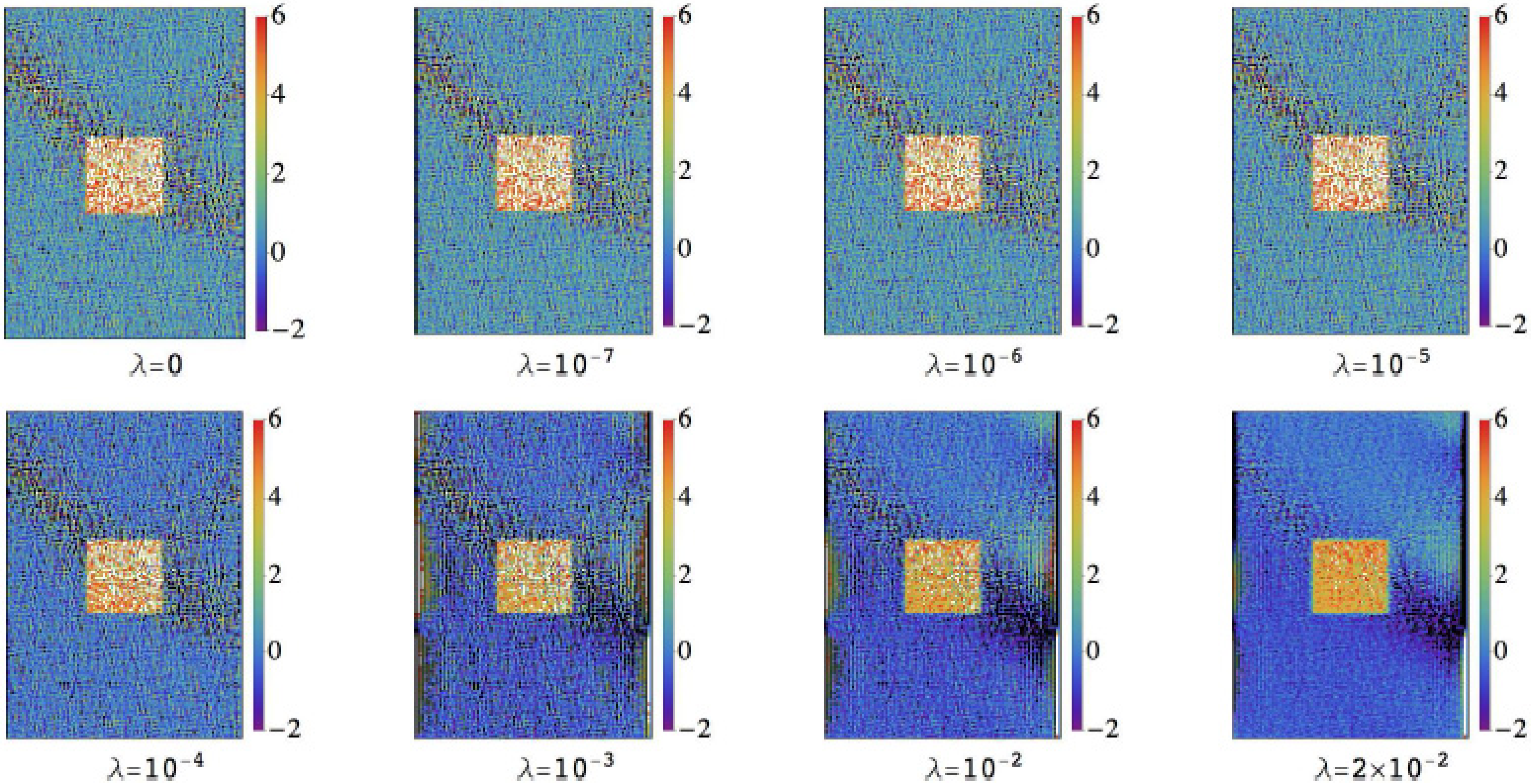
(b) Case 3b

CASE 3a; K=3, s1=s2=1, s3=-2,NZ=2; effects of regularization for N=1e4



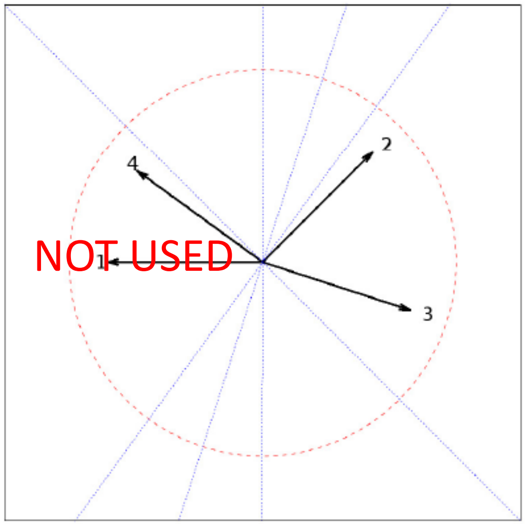
(a) Case 3a

CASE 3b; K=3, s1=s2=1, s3=-2,NZ=0; effects of regularization for N=1e4

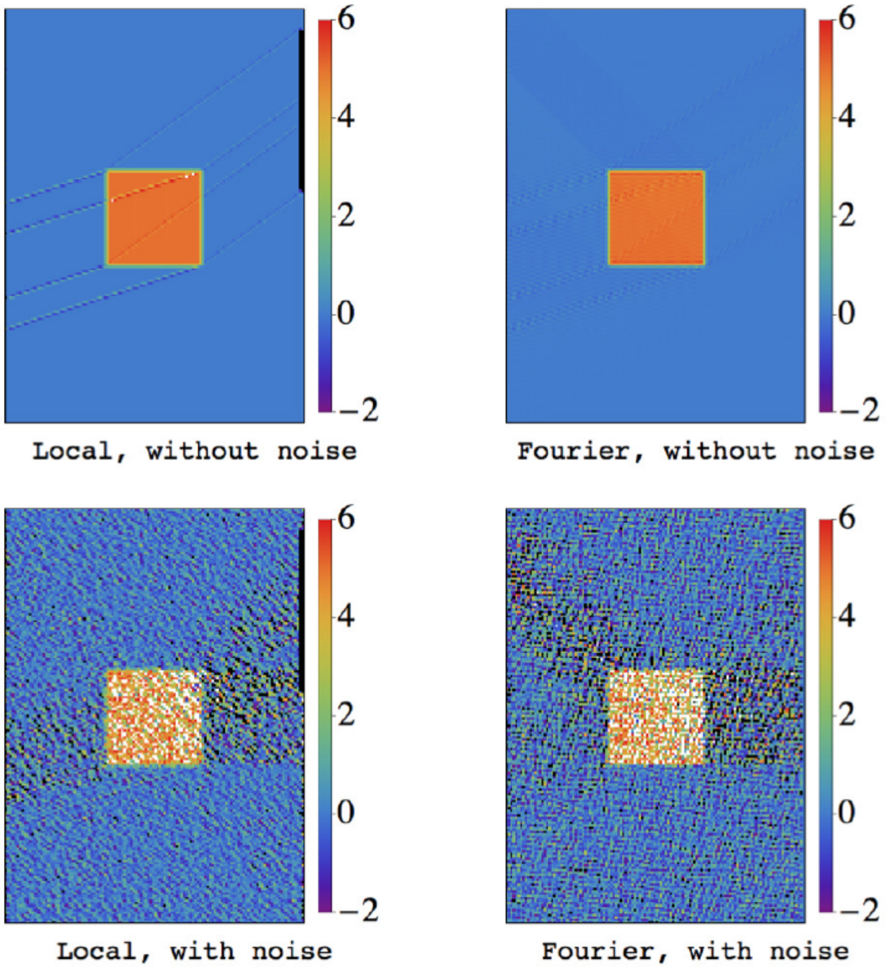


(b) Case 3b

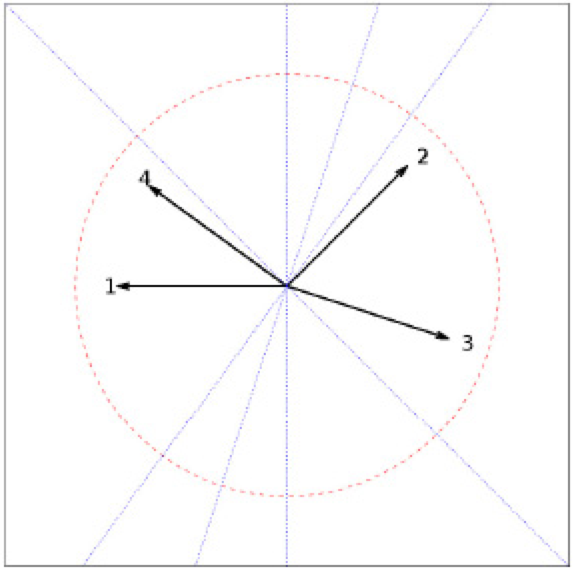
Comparison of local and Fourier methods (K=3 coefficient matrix)



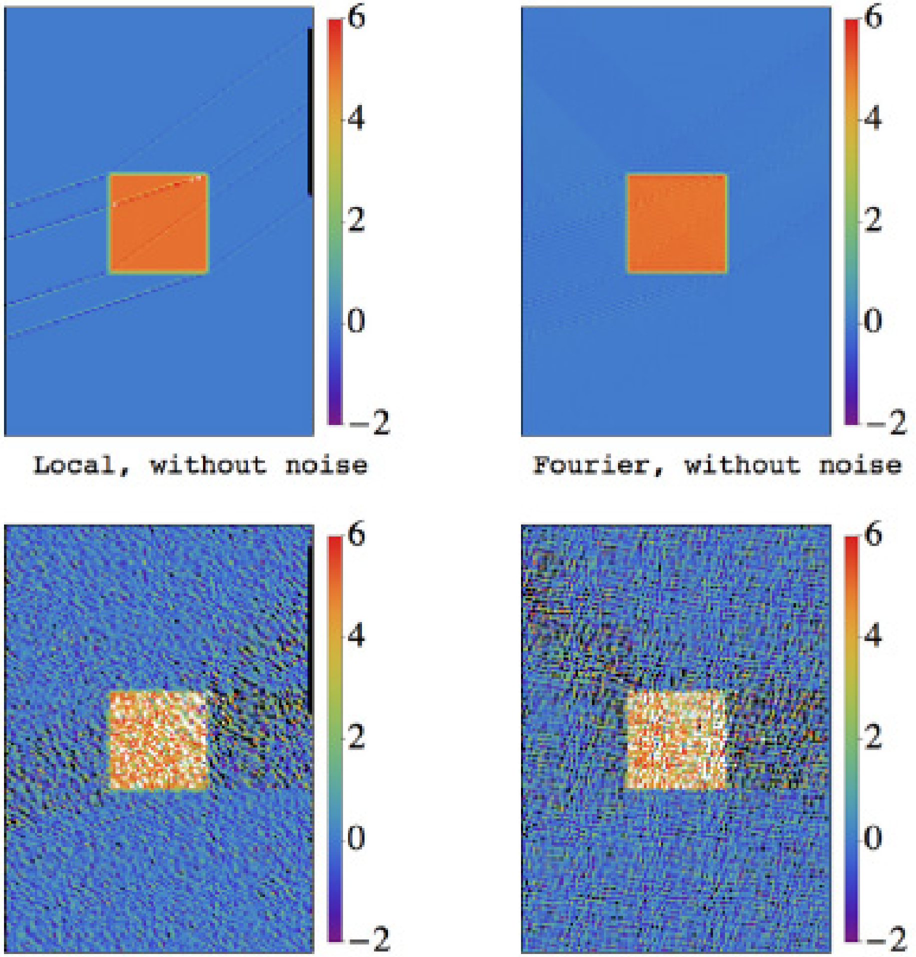
	α_k/π	c_k	s_k	σ_k
1	1.0	NOT USED		
2	0.25	1	1	−0.3468
3	−0.1	−2	−2	1.1085
4	0.8	1	1	1.0000



Comparison of local and Fourier methods (K=4, geometry of Katsevich and Krylov, first ray Canceled in the star transform)



	α_k/π	c_k	s_k	σ_k
1	1.0			
2	0.25	1	1	−0.3468
3	−0.1	−2	−2	1.1085
4	0.8	1	1	1.0000



Local, without noise

Fourier, without noise

Local, with noise

Fourier, with noise