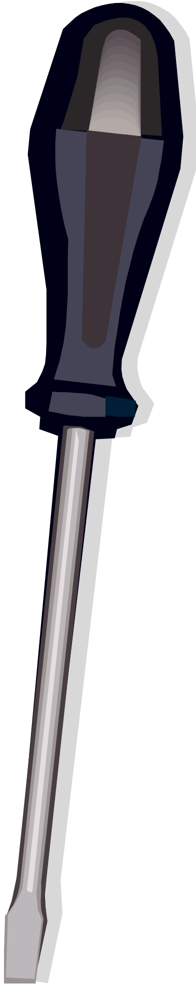
Tomography of Highly Scattering Media with the Method of Rotated Reference Frames



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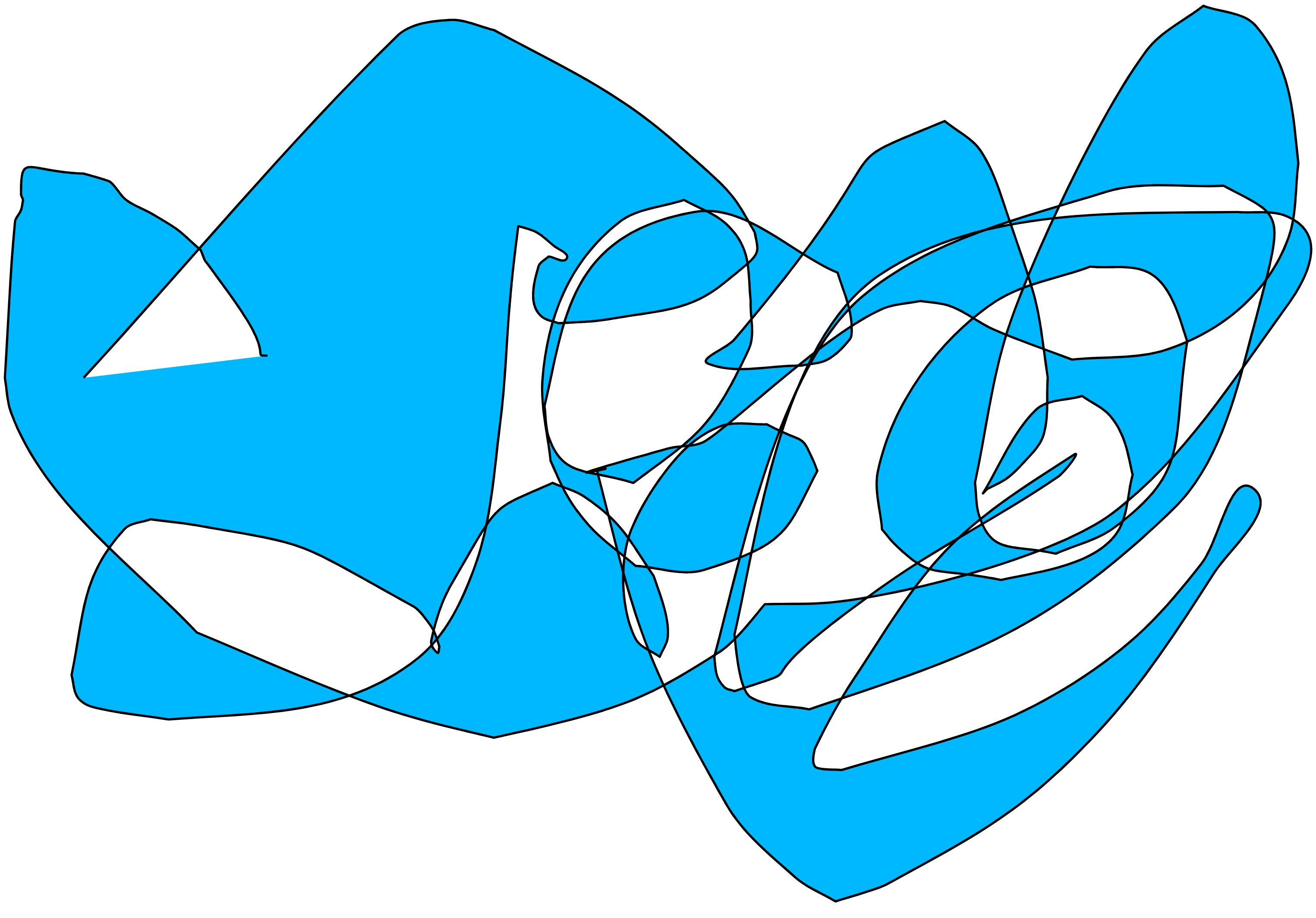
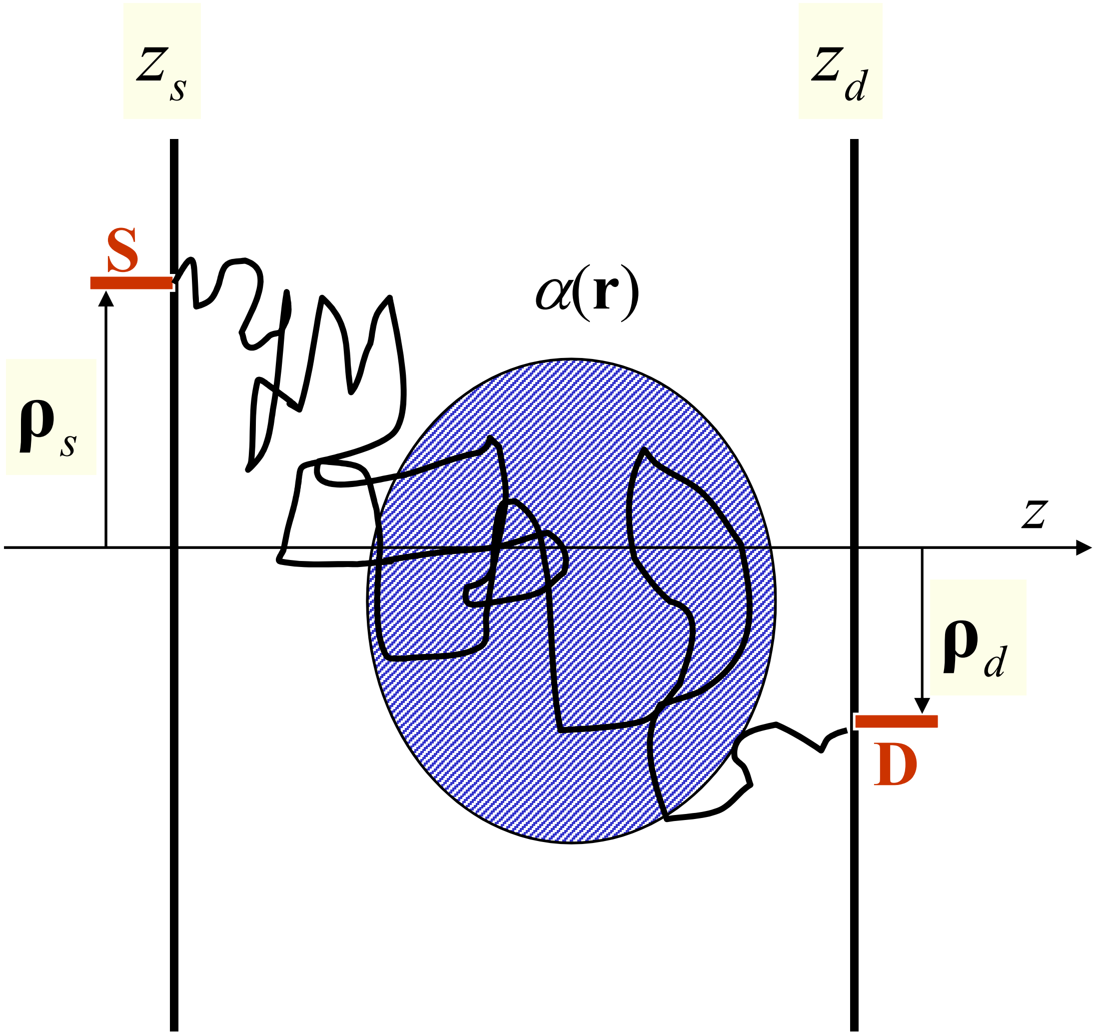
Outline



Motivation
(The Forward Problem Perspective)

	Plane Wave Modes	The Weyl Expanson
The Helmholtz Equation $(\nabla^2 + k_0^2)u = 0$ $k = \text{const}$	$e^{i\mathbf{k}\cdot\mathbf{r}}$ $\mathbf{k}\cdot\mathbf{k} = k_0^2$	$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi}\int Q^{-1} e^{i(\mathbf{q}\cdot\boldsymbol{\rho} + Qz)} d^2q$ $Q = \sqrt{k_0^2 - q^2}$
The Diffusion Equation $(-\nabla \cdot D\nabla + \alpha)u = 0$ $D, \alpha = \text{const}$	$e^{-\mathbf{k}\cdot\mathbf{r}}$ $\mathbf{k}\cdot\mathbf{k} = k_0^2 = \alpha / D$	$\frac{e^{-k_0 r}}{r} = \frac{1}{2\pi}\int Q^{-1} e^{i\mathbf{q}\cdot\boldsymbol{\rho} - Qz} d^2q$ $Q = \sqrt{k_0^2 + q^2}$
RTE $(\hat{\mathbf{s}}\cdot\nabla + \mu_t)I(\mathbf{r},\hat{\mathbf{s}}) =$ $= \mu_s \int A(\hat{\mathbf{s}},\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}')$ $\mu_t, \mu_s = \text{const}$	Not known	Not known

MOTIVATION
(The Inverse Problems Perspective)



Given a data function $\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d)$ which is measured for multiple pairs $(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d)$, find the absorption coefficient $\alpha(\mathbf{r})$ inside the slab

Linearized Integral Equation

$$\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = \int \Gamma(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d; \mathbf{r})\delta\alpha(\mathbf{r})\mathrm{d}^3 r$$

$$\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = \frac{I(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d) - I_0(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d)}{I_0(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d)}$$

(measurable data-function)

$$\Gamma(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = G_0(\boldsymbol{\rho}_s, z_s; \mathbf{r})G_0(\mathbf{r}; \boldsymbol{\rho}_d, z_d)$$

(first Born approximation)

$$\alpha(\mathbf{r}) = \alpha_0 + \delta\alpha(\mathbf{r})$$

Analytical SVD approach: Making use of the translational invariance

$$\tilde{\phi}(\mathbf{q}_s,\mathbf{q}_d) = \int \phi(\boldsymbol{\rho}_s,\boldsymbol{\rho}_d) e^{i(\mathbf{q}_s\cdot\boldsymbol{\rho}_s+\mathbf{q}_d\cdot\boldsymbol{\rho}_d)} \mathrm{d}^2\rho_s \mathrm{d}^2\rho_d$$

$$\mathbf{q}_s = \mathbf{q}/2+\mathbf{p}\ , \qquad \mathbf{q}_d = \mathbf{q}/2-\mathbf{p};$$

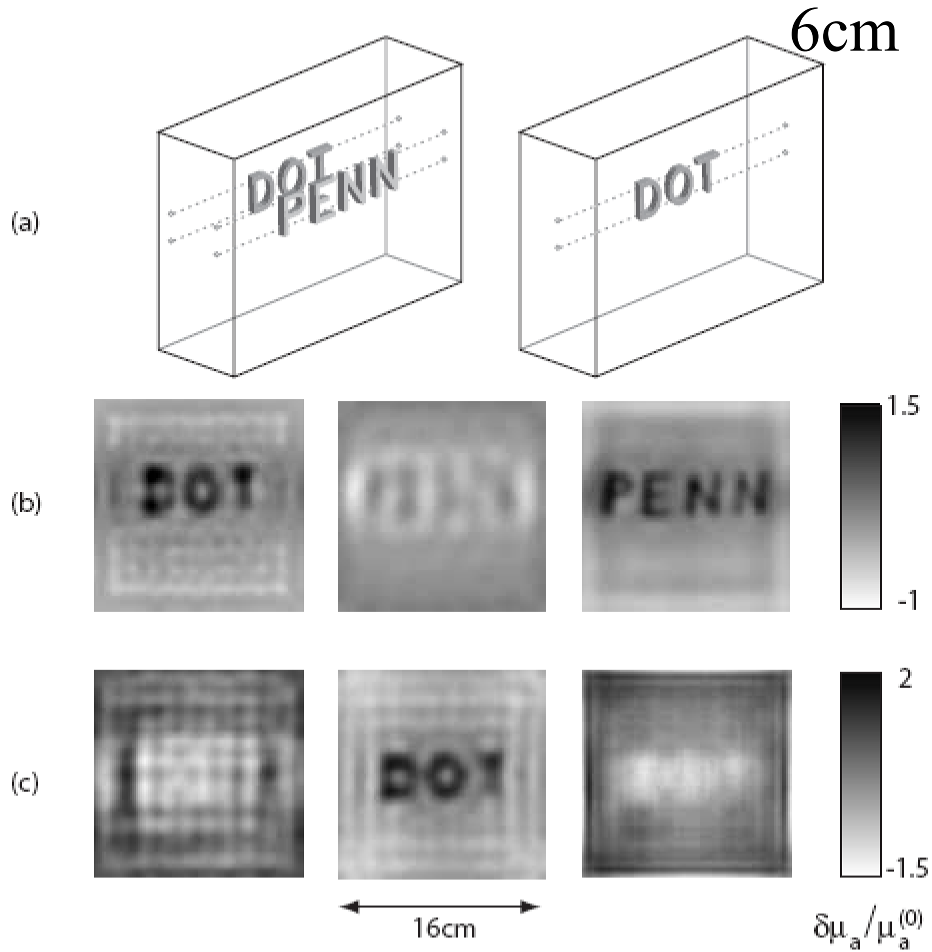
Data function: $\psi(\mathbf{q},\mathbf{p}) = \tilde{\phi}(\mathbf{q}/2+\mathbf{p},\mathbf{q}/2-\mathbf{p})$

$$\psi(\mathbf{q},\mathbf{p}) = \int_0^L g_s(\mathbf{q}/2+\mathbf{p};z) g_d(\mathbf{q}/2-\mathbf{p};z) \delta\tilde{\alpha}(\mathbf{q};z)\mathrm{d}z$$

$$\delta\tilde{\alpha}(\mathbf{q};z)dz = \int \delta\alpha(\boldsymbol{\rho},z) e^{i\boldsymbol{\rho}\cdot\mathbf{q}} \mathrm{d}^2\rho$$

$$G_0(\boldsymbol{\rho}_s,z_s;\boldsymbol{\rho},z) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g_s(\mathbf{q};z) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_s)}$$

$$G_0(\boldsymbol{\rho},z;\boldsymbol{\rho}_d,z_d) = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g_d(\mathbf{q};z) e^{i\mathbf{q}\cdot(\boldsymbol{\rho}_d-\boldsymbol{\rho})}$$



S.D.Konecky, G.Y.Panasyuk, K.Lee, V.Markel, A.G.Yodh and J.C.Schotland
Imaging complex structures with diffuse light
Optics Express **16(7), 5048-5060 (2008)**

In the case of RTE, we need the plane-wave decomposition of the Greens function, of the form

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}') = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g(\mathbf{q}; z,\hat{\mathbf{s}}; z',\hat{\mathbf{s}}') e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}$$

and the integral kernel

$$\Gamma(\mathbf{q},\mathbf{p};z) = \int g(\mathbf{q}/2+\mathbf{p}; z_s,\hat{\mathbf{z}}; z,\hat{\mathbf{s}}) g(\mathbf{q}/2-\mathbf{p}; z,\hat{\mathbf{s}}; z_d,\hat{\mathbf{z}}) \mathrm{d}^2 s$$

We then get the integral equations of the form

$$\psi(\mathbf{q},\mathbf{p}) = \int_{z_s}^{z_d} \Gamma(\mathbf{q},\mathbf{p};z)\delta\tilde{\alpha}(\mathbf{q};z)\mathrm{d}z$$

THE METHOD

Rotated Reference Frames

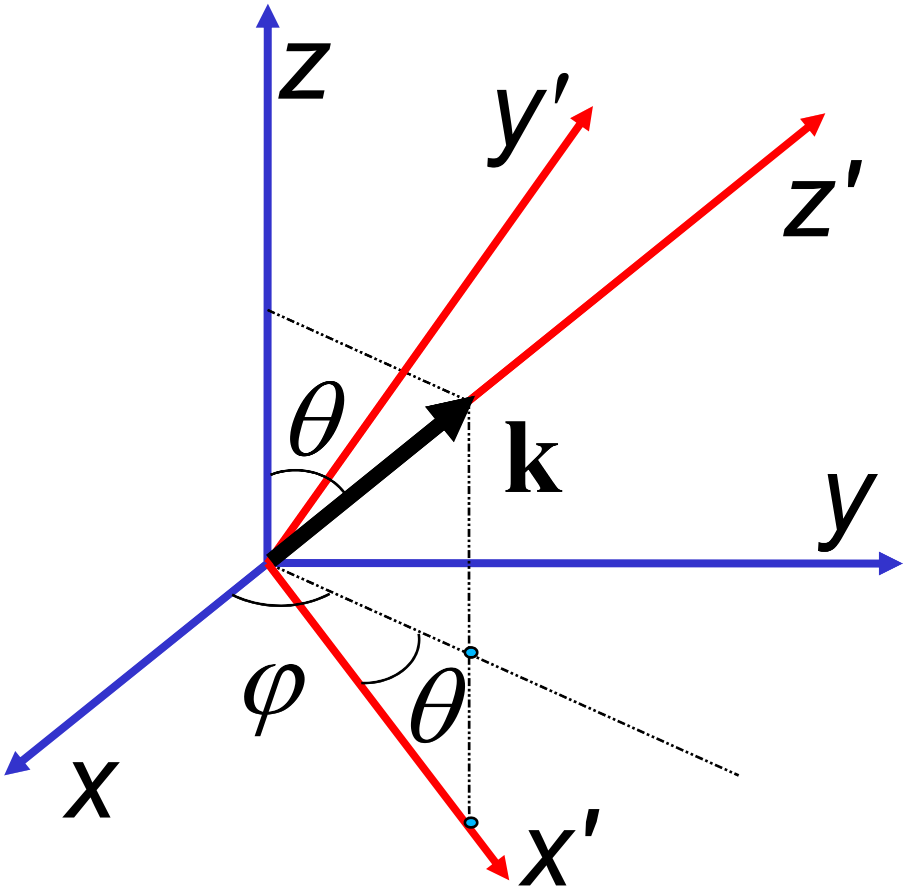
The usual sperical harmonics are defined in the laboratory reference frame. Then θ and φ are the polar angles of the unit vector $\hat{\mathbf{s}}$ in that frame.

THE MAIN IDEA:

For each value of the Fourier variable $\mathbf{k}$, use spherical harmonics defined in a reference frame whose z-axis is aligned with the direction of $\mathbf{k}$.

We call such frames "rotated".
Spherical harmonics defined in the rotaded frame are denoted by $Y(\hat{\mathbf{s}};\hat{\mathbf{k}})$.

Rotation of the Laboratory Frame (x,y,z).



$$Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}}) = \sum_{m'=-l}^{l} D^{l}_{m'm}(\varphi_{\mathbf{k}},\theta_{\mathbf{k}},0)Y_{lm'}(\hat{\mathbf{s}})$$

Wigner D-functions

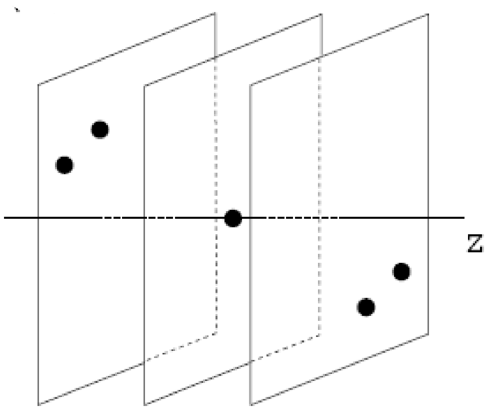
Euler angles

Spherical functions in the laboratory frame

Advantages of the Method

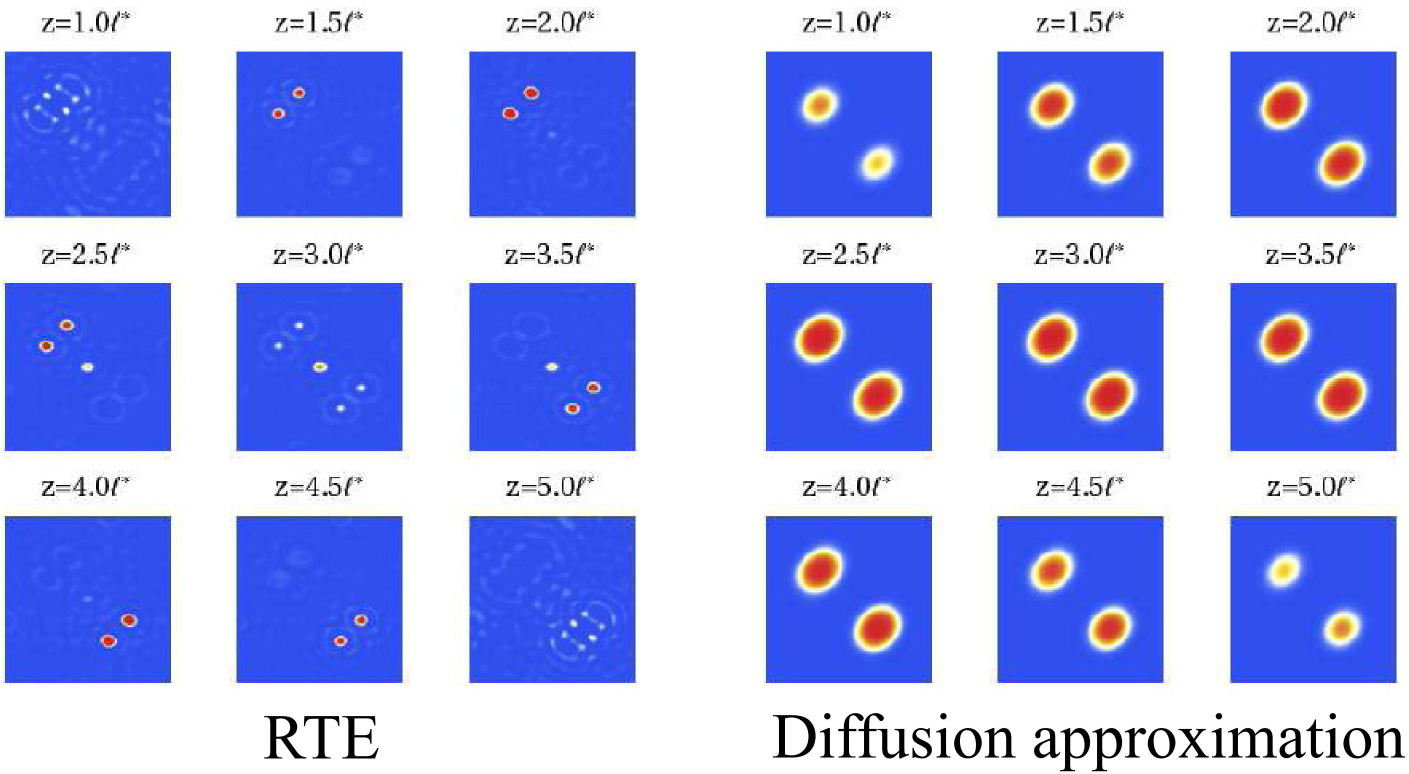
- Numerical problem is reduced to diagonalization of a set of tridiagonal matrices
- Once the eigenvalues and eigenvectors are computed, the Green's function, the plane-wave modes and the Weyl expansion for the GF can be obtained in analytical form
- The Weyl expansion can also be obtained in a slab with appropriate boundary conditions

RESULTS: SIMULATIONS

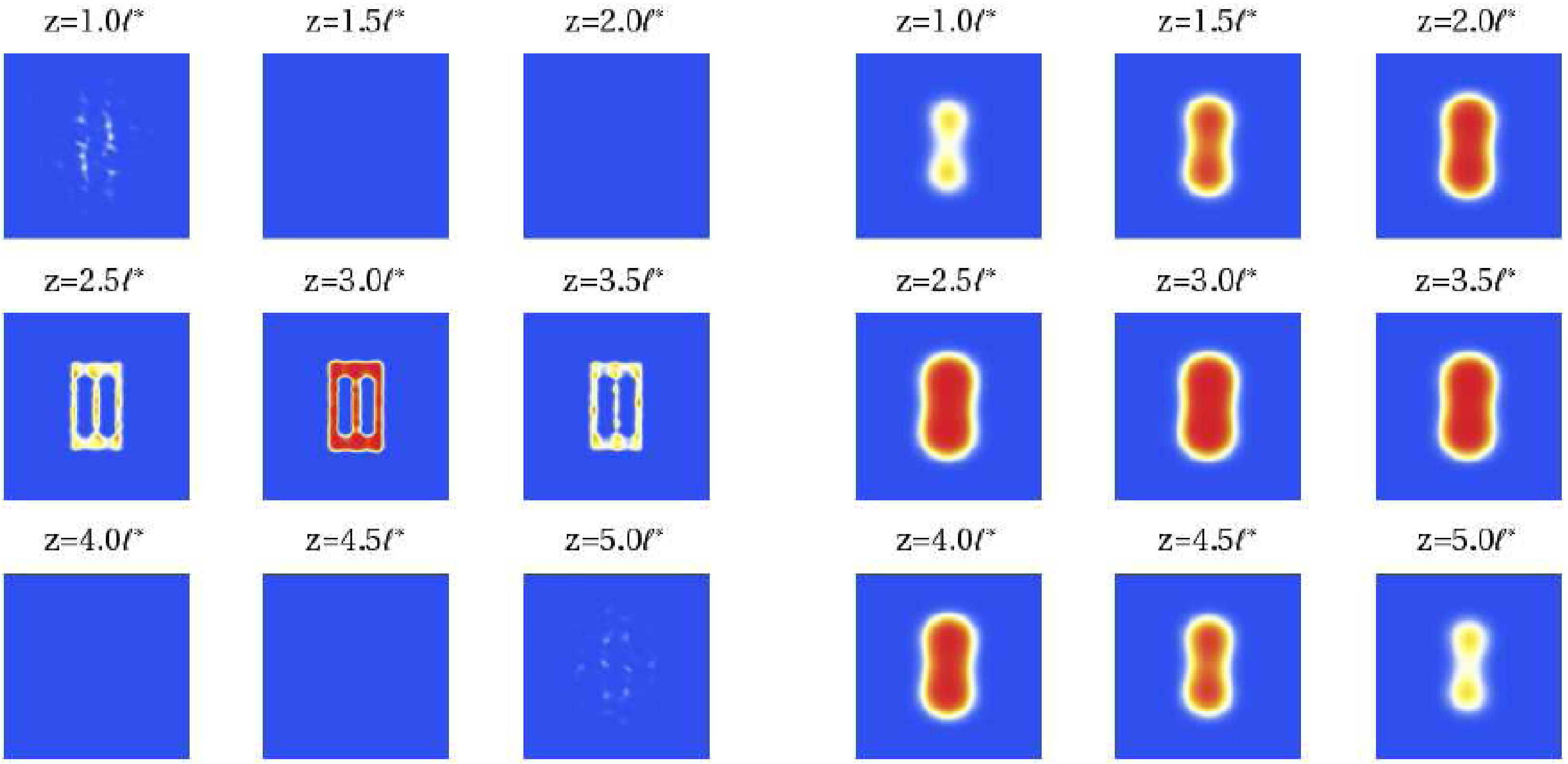


A set of 5 point absorbers in an L=6ℓ^* slab

The field of view is 16ℓ^*



A bar target in the center of the same slab

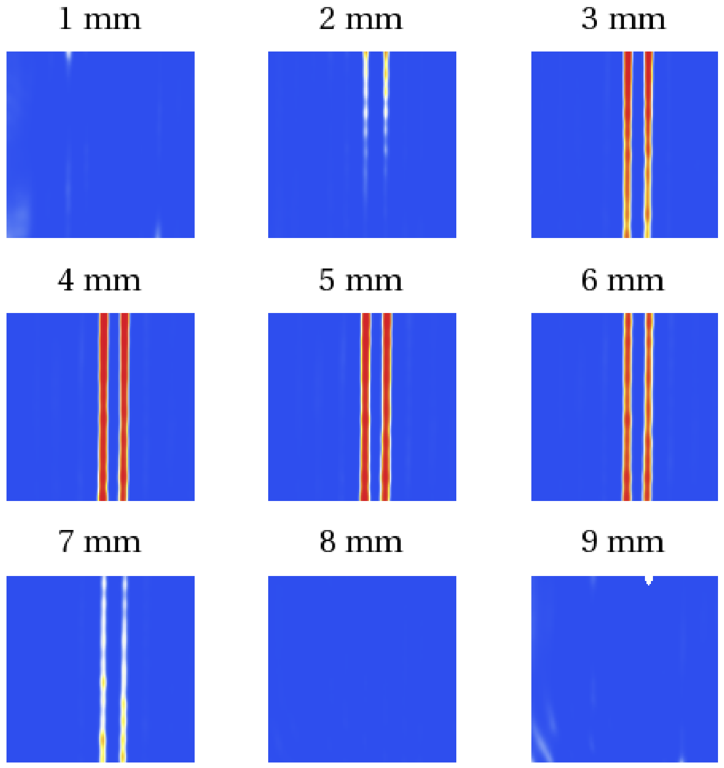


RTE

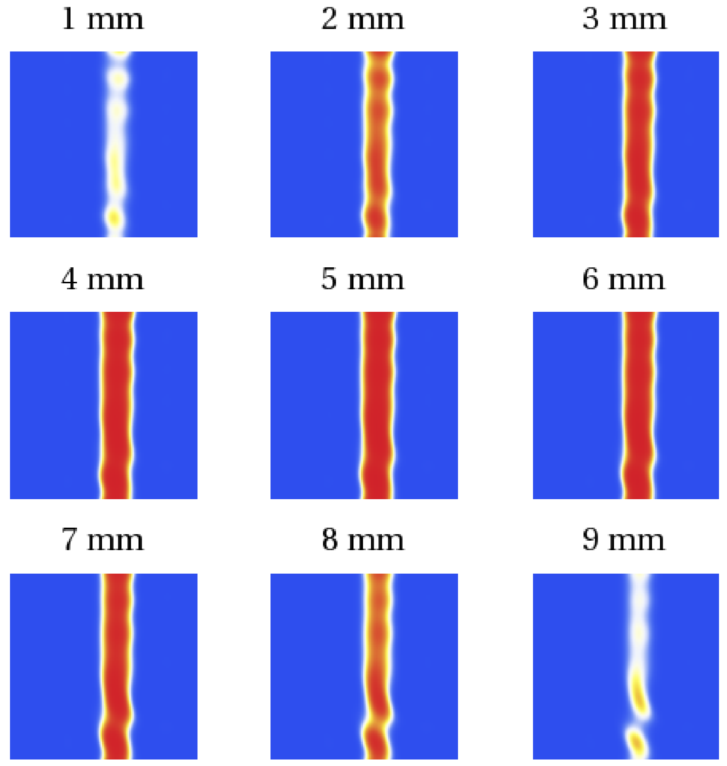
Diffusion approximation

RESULTS: SIMULATIONS

Two thin vertical wires in a 1cm thick slab filled with intralipid solution (looks like milk)



RTE



Diffusion approximation

Publications:

1. V.A.Markel, “Modified spherical harmonics method for solving the radiative transport equation,” *Waves in Random Media* **14**(1), L13-L19 (2004).

2. G.Y.Panasyuk, J.C.Schotland, and V.A.Markel, “Radiative transport equation in rotated reference frames,” *Journal of Physics A*, **39**(1), 115-137 (2006).

3. J.C.Schotland and V.A.Markel , “Fourier-Laplace structure of the inverse scattering problem for the radiative transport equation,” Inverse Problems and Imaging 1(1), 181-188 (2007).

Available on the web at
http://whale.seas.upenn.edu/vmarkel/papers.html

CONCLUSIONS

- The method of rotated reference frames can be used in optical tomography of mesoscopic samples
- The images are of superior quality compared to those obtained by using the diffusion approximation
- The quality of images can be comparable to that in X-ray tomography because the RTE retains some information about ballistic rays, single-scattered rays, etc.