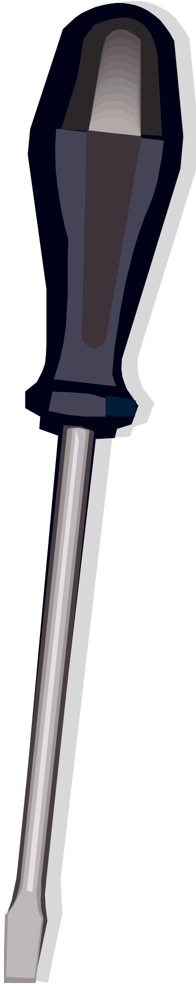
Tomography of Highly Scattering Media with the Method of Rotated Reference Frames



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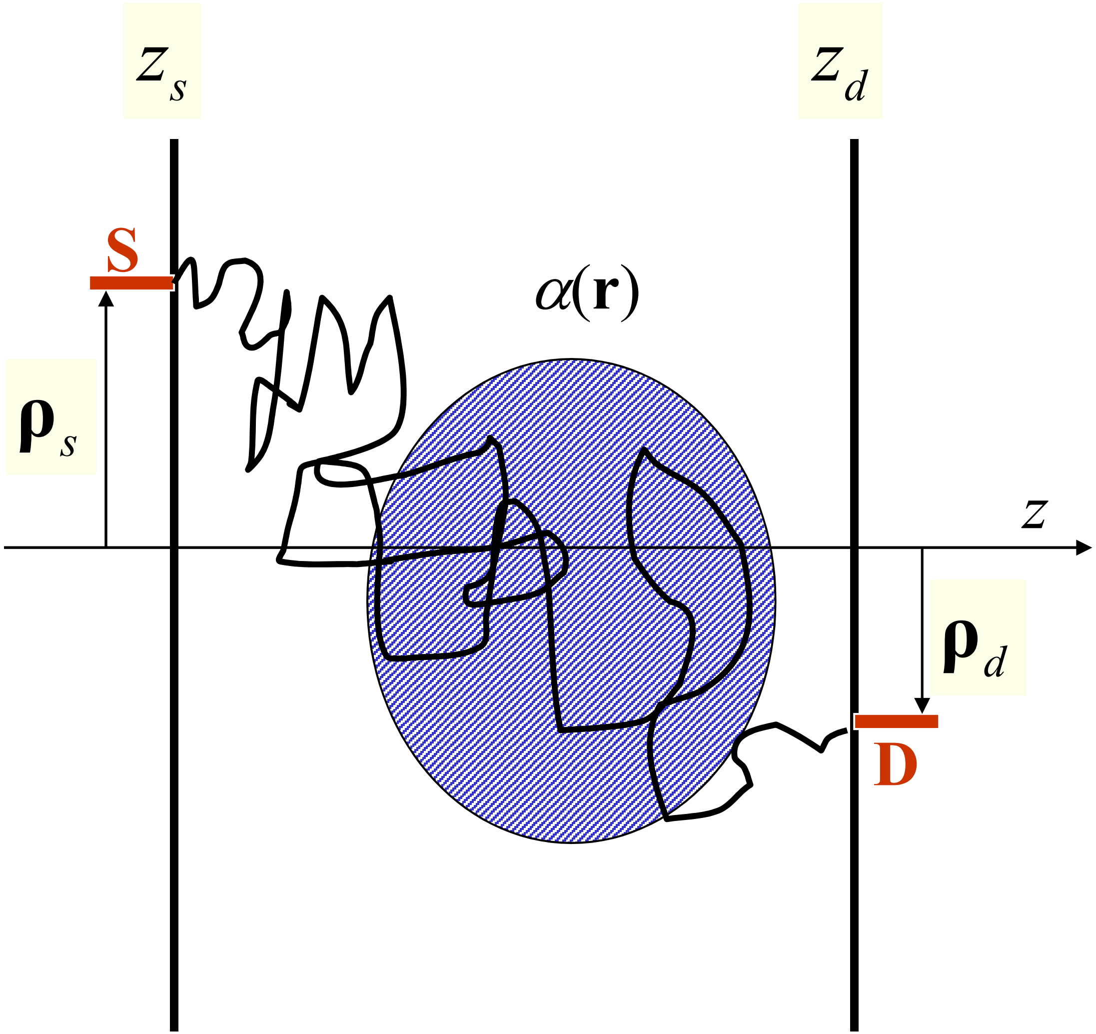
MOTIVATION
(The Inverse Problem Perspective)



Given a data function $\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d)$ which is measured for multiple pairs $(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d)$, find the absorption coefficient $\alpha(\mathbf{r})$ inside the slab

Linearized Integral Equation

$$\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = \int \Gamma(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d; \mathbf{r}) \delta\alpha(\mathbf{r}) \mathrm{d}^3 r$$

$$\phi(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = \frac{I(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d) - I_0(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d)}{I_0(\boldsymbol{\rho}_s, z_s; \boldsymbol{\rho}_d, z_d)}$$

(measurable data-function)

$$\Gamma(\boldsymbol{\rho}_s, \boldsymbol{\rho}_d) = G_0(\boldsymbol{\rho}_s, z_s; \mathbf{r}) G_0(\mathbf{r}; \boldsymbol{\rho}_d, z_d)$$

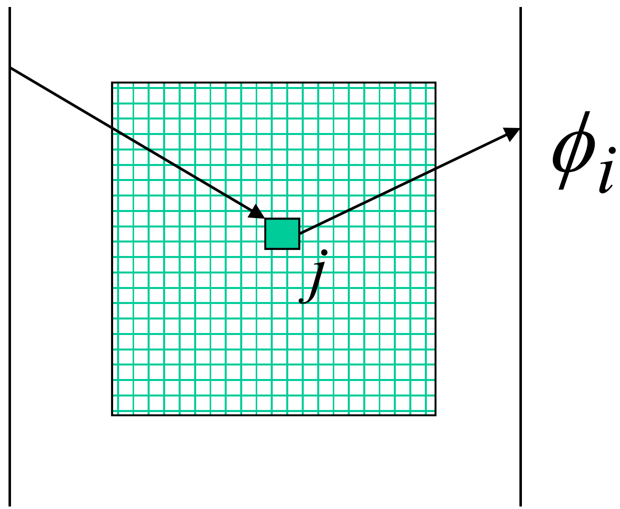
(first Born approximation)

$$\alpha(\mathbf{r}) = \alpha_0 + \delta\alpha(\mathbf{r})$$

How To Invert?

Numerical SVD approach:

$$\phi_i = \sum_j \Gamma_{ij} \delta\alpha_j$$



$$\phi = \Gamma \delta\alpha$$

$$\Gamma^* \phi = \Gamma^* \Gamma \delta\alpha$$

$$\delta\alpha^+ = \left(\Gamma^* \Gamma\right)^{-1}_{reg} \Gamma^* \phi$$

Analytical SVD approach: Making use of the translational invariance

$$\tilde{\phi}(\mathbf{q}_s,\mathbf{q}_d)=\int\phi(\boldsymbol{\rho}_s,\boldsymbol{\rho}_d)e^{i(\mathbf{q}_s\cdot\boldsymbol{\rho}_s+\mathbf{q}_d\cdot\boldsymbol{\rho}_d)}\mathrm{d}^2\rho_s\mathrm{d}^2\rho_d$$

$$\mathbf{q}_s=\mathbf{q}/2+\mathbf{p}\ ,\qquad \mathbf{q}_d=\mathbf{q}/2-\mathbf{p};$$

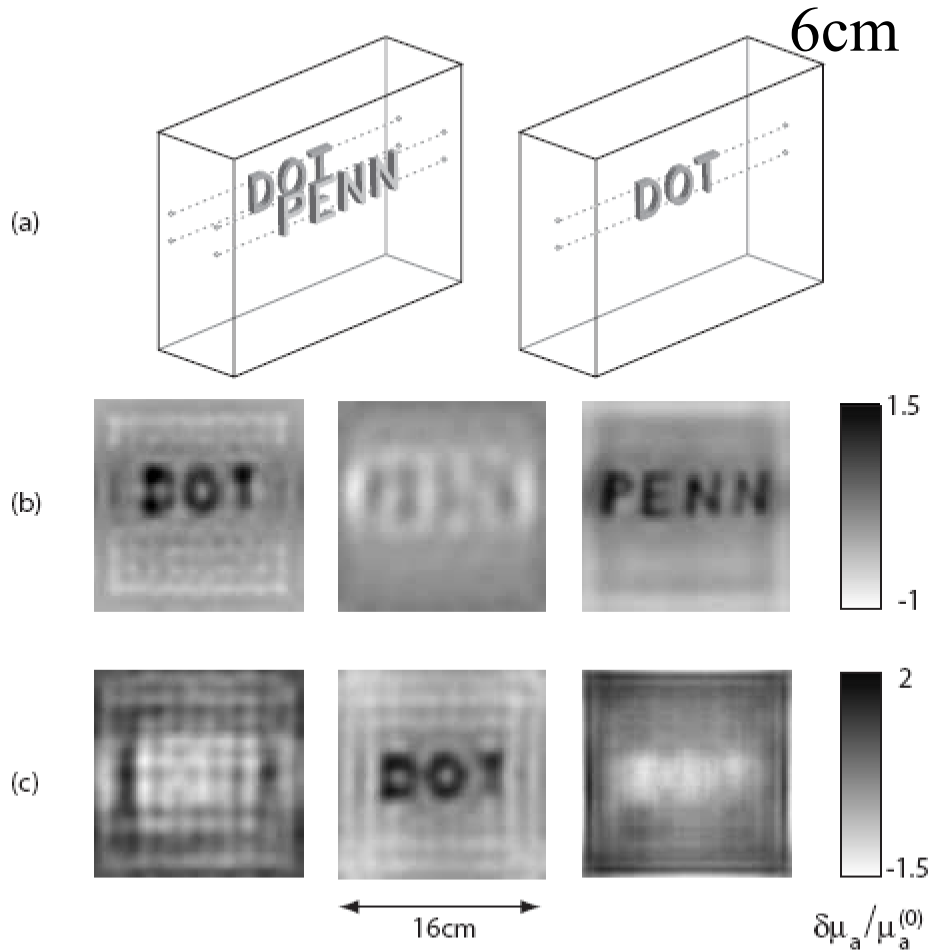
Data function: $\psi(\mathbf{q},\mathbf{p})=\tilde{\phi}(\mathbf{q}/2+\mathbf{p},\mathbf{q}/2-\mathbf{p})$

$$\psi(\mathbf{q},\mathbf{p})=\int_0^L g_s(\mathbf{q}/2+\mathbf{p};z)g_d(\mathbf{q}/2-\mathbf{p};z)\delta\tilde{\alpha}(\mathbf{q};z)\mathrm{d}z$$

$$\delta\tilde{\alpha}(\mathbf{q};z)dz=\int\delta\alpha(\boldsymbol{\rho},z)e^{i\boldsymbol{\rho}\cdot\mathbf{q}}\mathrm{d}^2\rho$$

$$G_0(\boldsymbol{\rho}_s,z_s;\boldsymbol{\rho},z)=\int\frac{\mathrm{d}^2q}{(2\pi)^2}g_s(\mathbf{q};z)e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_s)}$$

$$G_0(\boldsymbol{\rho},z;\boldsymbol{\rho}_d,z_d)=\int\frac{\mathrm{d}^2q}{(2\pi)^2}g_d(\mathbf{q};z)e^{i\mathbf{q}\cdot(\boldsymbol{\rho}_d-\boldsymbol{\rho})}$$



S.D.Konecky, G.Y.Panasyuk, K.Lee, V.Markel, A.G.Yodh and J.C.Schotland
Imaging complex structures with diffuse light
Optics Express **16(7), 5048-5060 (2008)**

In the case of RTE, we need the plane-wave decomposition of the Greens function, of the form

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}',\hat{\mathbf{s}}') = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} g(\mathbf{q};z,\hat{\mathbf{s}};z',\hat{\mathbf{s}}') e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}')}$$

and the integral kernel

$$\Gamma(\mathbf{q},\mathbf{p};z) = \int g(\mathbf{q}/2+\mathbf{p};z_s,\hat{\mathbf{z}};z,\hat{\mathbf{s}}) g(\mathbf{q}/2-\mathbf{p};z,\hat{\mathbf{s}};z_d,\hat{\mathbf{z}}) \mathrm{d}^2 s$$

We then get the integral equations of the form

$$\psi(\mathbf{q},\mathbf{p}) = \int_{z_s}^{z_d} \Gamma(\mathbf{q},\mathbf{p};z) \delta\tilde{\alpha}(\mathbf{q};z) \mathrm{d}z$$

Motivation Continued
(The Forward Model Perspective)

	Plane Wave Modes	The Weyl Expanson
The Helmholtz Equation $(\nabla^2 + k_0^2)u = 0$ $k = \mathrm{const}$	$e^{i\mathbf{k}\cdot\mathbf{r}}$ $\mathbf{k}\cdot\mathbf{k} = k_0^2$	$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi}\int Q^{-1} e^{i(\mathbf{q}\cdot\boldsymbol{\rho}+Qz)} d^2q$ $Q = \sqrt{k_0^2 - q^2}$
The Diffusion Equation $(-D\nabla^2 + \alpha)u = 0$ $D, \alpha = \mathrm{const}$	$e^{-\mathbf{k}\cdot\mathbf{r}}$ $\mathbf{k}\cdot\mathbf{k} = k_0^2 = \alpha / D$	$\frac{e^{-k_0 r}}{r} = \frac{1}{2\pi}\int Q^{-1} e^{i\mathbf{q}\cdot\boldsymbol{\rho}-Qz} d^2q$ $Q = \sqrt{k_0^2 + q^2}$
RTE $(\hat{\mathbf{s}}\cdot\nabla + \mu_t)I(\mathbf{r},\hat{\mathbf{s}}) =$ $= \mu_s \int A(\hat{\mathbf{s}},\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}')$ $\mu_t, \mu_s = \mathrm{const}$	Arnold Kim and us (the past ~10 years)	Arnold Kim and us (the past ~10 years)

Spectral Method for the RTE ?

RTE: $(\hat{\mathbf{s}}\cdot\nabla+\mu_t)I(\mathbf{r},\hat{\mathbf{s}})=\mu_s\int A(\hat{\mathbf{s}},\hat{\mathbf{s}}')I(\mathbf{r},\hat{\mathbf{s}}')\mathrm{d}^2\hat{\mathbf{s}}'+\varepsilon(\mathbf{r},\hat{\mathbf{s}})$

Where is the "spectral variable" ?

How can we write this equation in the form $(z+W)|I\rangle=|\varepsilon\rangle$?

μ_t and μ_s do not qualify...

We can try to expand $I(\mathbf{r},\hat{\mathbf{s}})$ into a 3D Fourier integral with respect to $\mathbf{r}$ and into the basis of ordinary spherical harmonics $Y_{lm}(\theta,\varphi)$ with respect to $\hat{\mathbf{s}}$ ...

...and see if the equation can be cast into the desired form.

The Conventional Method of Spherical Harmonics

This results in the following system of equations with respect to the vector of the expansion coefficients $\left|I(\mathbf{k})\right\rangle$ (**k** - the Fourier variable) :

$$iA^{(x)}k_x\left|I(\mathbf{k})\right\rangle + iA^{(y)}k_y\left|I(\mathbf{k})\right\rangle + iA^{(z)}k_z\left|I(\mathbf{k})\right\rangle + D\left|I(\mathbf{k})\right\rangle = \left|\varepsilon(\mathbf{k})\right\rangle$$

$A^{(x)}, \quad A^{(y)}, \quad A^{(z)}$ are different matrices.

$$A^{(x)}_{lm,l'm'} = \int \sin\theta\cos\varphi Y^*_{lm}(\theta,\varphi)Y_{l'm'}(\theta,\varphi)\sin\theta d\theta d\varphi \quad , \qquad \text{etc. .....}$$

“This rather awe-inspiring set of equations … has perhaps only academic interest”.

K.M.Case, P.F.Zweifel, Linear Transport Theory

Rotated Reference Frames

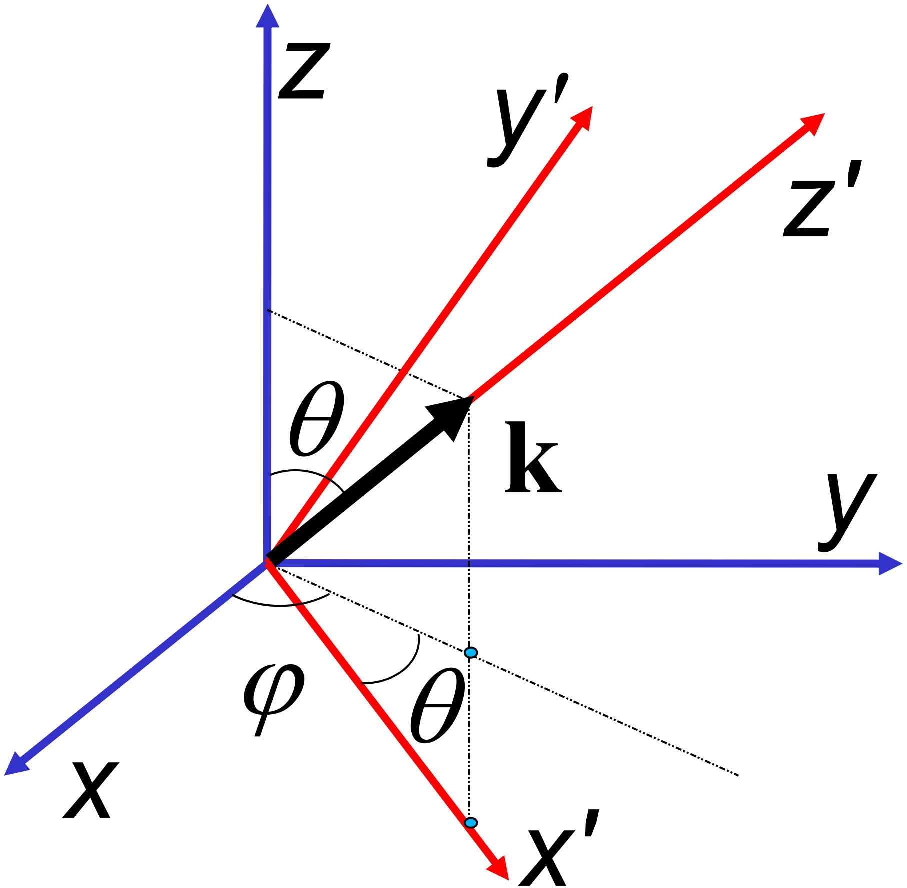
The usual sperical harmonics are defined in the laboratory reference frame. Then θ and φ are the polar angles of the unit vector $\hat{\mathbf{s}}$ in that frame.

THE MAIN IDEA:

For each value of the Fourier variable $\mathbf{k}$, use spherical harmonics defined in a reference frame whose z-axis is aligned with the direction of $\mathbf{k}$.

We call such frames "rotated".
Spherical harmonics defined in the rotaded frame are denoted by $Y(\hat{\mathbf{s}};\hat{\mathbf{k}})$.

Rotation of the Laboratory Frame (x,y,z).



$$Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}}) = \sum_{m'=-l}^{l} D^{l}_{m'm}(\varphi_{\mathbf{k}},\theta_{\mathbf{k}},0)\, Y_{lm'}(\hat{\mathbf{s}})$$

Wigner D-functions

Euler angles

Spherical functions in the laboratory frame

$$(\hat{\mathbf{s}} \cdot \nabla + \mu_t) I = \mu_s A I + \varepsilon$$

$$I(\mathbf{r}, \hat{\mathbf{s}}) = \int \tilde{I}(\mathbf{k}, \hat{\mathbf{s}}) e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 k$$

$$\varepsilon(\mathbf{r}, \hat{\mathbf{s}}) = \int \tilde{\varepsilon}(\mathbf{k}, \hat{\mathbf{s}}) e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3 k$$

$$(i\mathbf{k} \cdot \hat{\mathbf{s}} + \mu_t) \tilde{I} = \mu_s A \tilde{I} + \tilde{\varepsilon}$$

$$\tilde{I}(\mathbf{k}, \hat{\mathbf{s}}) = \sum_{l,m} F_{lm}(\mathbf{k}) Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}})$$

$$\tilde{\varepsilon}(\mathbf{k}, \hat{\mathbf{s}}) = \sum_{l,m} E_{lm}(\mathbf{k}) Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}})$$

$$A(\hat{\mathbf{s}}, \hat{\mathbf{s}}') = \sum_{l,m} A_l Y_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}}) Y^*_{lm}(\hat{\mathbf{s}}; \hat{\mathbf{k}})$$

$$ik\sum_{l'm'} R_{lm}^{l'm'} F_{l'm'}(\mathbf{k}) + \sigma_l F_{lm}(\mathbf{k}) = E_{lm}(\mathbf{k})$$

$$\sigma_l = \mu_a + \mu_s(1 - A_l)$$

$$R_{lm}^{l'm'} = \int \hat{\mathbf{s}} \cdot \hat{\mathbf{k}} \; Y_{lm}^*(\hat{\mathbf{s}};\hat{\mathbf{k}}) Y_{l'm'}(\hat{\mathbf{s}};\hat{\mathbf{k}}) \mathrm{d}^2 s =$$

$$= \delta_{mm'} \left[b_l(m) \delta_{l'=l-1} + b_{l+1}(m) \delta_{l'=l+1} \right]$$

$$b_l(m) = \sqrt{\frac{l^2 - m^2}{4l^2 - 1}}$$

RTE in the Angular Basis of Rotated Spherical Functions

$$ikR\left|I(\mathbf{k})\right\rangle + D\left|I(\mathbf{k})\right\rangle = \left|E(\mathbf{k})\right\rangle$$

Source term

Scalar spectral parameter

Block-tridiagonal real symmetric matrix

Diagonal matrix
$S_{lm,l'm'} = \delta_{ll'}\delta_{mm'}[\mu_a + \mu_s(1 - A_l)]$

Parameters of the phase function. (For the HG model, $A_l = g^l$)

Let $D = SS$

$$W = S^{-1} R S^{-1}$$

$$ikR \left| I(\mathbf{k}) \right\rangle + D \left| I(\mathbf{k}) \right\rangle = \left| E(\mathbf{k}) \right\rangle$$

$$(ikW + 1) S \left| I(\mathbf{k}) \right\rangle = S^{-1} \left| E(\mathbf{k}) \right\rangle$$

$$\left| I(\mathbf{k}) \right\rangle = S^{-1} (1 + ikW)^{-1} S^{-1} \left| E(\mathbf{k}) \right\rangle$$

$$W \left| \psi_\mu \right\rangle = \lambda_\mu \left| \psi_\mu \right\rangle$$

$$\left| I(\mathbf{k}) \right\rangle = \sum_\mu \frac{S^{-1} \left| \psi_\mu \right\rangle \left\langle \psi_\mu \right| S^{-1} \left| E(\mathbf{k}) \right\rangle}{1 + ik\lambda_\mu}$$

The Spectral Solution

$$I(\mathbf{r},\hat{\mathbf{s}}) = \sum_{lm} \frac{1}{\sqrt{\sigma_l}} \int \sum_{\mu} \frac{\langle lm|\psi_\mu\rangle\langle\psi_\mu|S^{-1}|E(\mathbf{k})\rangle Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}})}{1+ik\lambda_\mu} e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3k$$

$$\varepsilon(\mathbf{r},\hat{\mathbf{s}}) = \delta(\mathbf{r}-\mathbf{r}_0)\delta(\hat{\mathbf{s}}-\hat{\mathbf{s}}_0)$$

$$E_{lm}(\mathbf{k}) = \frac{1}{(2\pi)^3} e^{-i\mathbf{k}\cdot\mathbf{r}_0} Y^*_{lm}(\hat{\mathbf{s}}_0;\hat{\mathbf{k}})$$

The integral is not easy… but doable.

Details in J.Phys.A 39, 115 (2006)

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = \sum_{m=-\infty}^{\infty} \sum_{l,l'=|m|}^{\infty} Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{R}})\chi_{ll'}^{m}(R)Y_{l'm}^{*}(\hat{\mathbf{s}}_0;\hat{\mathbf{R}})$$

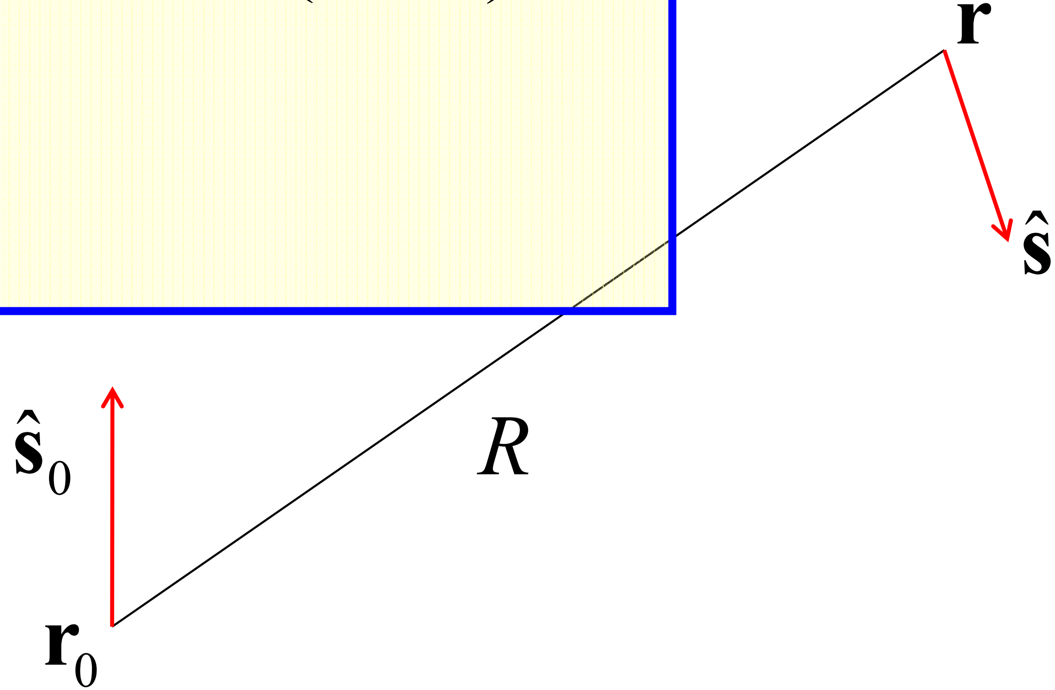
$$\chi_{ll'}^{m}(R) = \frac{(-1)^m}{2\pi\sqrt{\sigma_l \sigma_{l'}}} \sum_{M=-\overline{l}}^{\overline{l}} (-1)^M \sum_{n,\lambda_n>0} \frac{\langle l|\phi_n(M)\rangle\langle\phi_n(M)|l'\rangle}{\lambda_n^3(M)}$$

$$\times \sum_{j=0}^{\overline{l}} C_{l,M,l',-M}^{|l-l'|+2j,0} C_{l,m,l',-m}^{|l-l'|+2j,0} k_{|l-l'|+2j}\left(\frac{R}{\lambda_n(M)}\right)$$

$$\overline{l} = \min(l,l')$$

$$\mu = (M,n)$$

$$\langle lm|\psi_{Mn}\rangle = \delta_{mM}\langle l|\phi_n(M)\rangle$$



The Plane-Wave Decomposition

$$I(\mathbf{r},\hat{\mathbf{s}}) = \sum_{lm} \frac{1}{\sqrt{\sigma_l}} \int \sum_{\mu} \frac{\langle lm|\psi_\mu\rangle\langle\psi_\mu|S^{-1}|E(\mathbf{k})\rangle Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}})}{1+ik\lambda_\mu} e^{i\mathbf{k}\cdot\mathbf{r}} \mathrm{d}^3k$$

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = \sum_{lm,l'm'} \int \frac{d^2q}{(2\pi)^2} e^{i\mathbf{q}\cdot(\boldsymbol{\rho}-\boldsymbol{\rho}_0)} Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{z}}) g_{lm}^{l'm'}(\mathbf{q};z,z_0) Y_{l'm'}^{*}(\hat{\mathbf{s}}_0;\hat{\mathbf{z}})$$

$$g(\mathbf{q};z,z_0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k_z}{2\pi} e^{ik_z(z-z_0)} \mathscr{D}(\mathbf{q}+\hat{\mathbf{z}}k_z) K\left(\sqrt{q^2+k_z^2}\right) \mathscr{D}^{\dagger}(\mathbf{q}+\hat{\mathbf{z}}k_z)$$

$$K(w) = \sum_{\mu} \frac{S^{-1}|\psi_\mu\rangle\langle\psi_\mu|S^{-1}}{1+iw\lambda_\mu} \; ; \; \mathscr{D}(\hat{\mathbf{k}}) = e^{-i\varphi_{\mathbf{k}}J_z} e^{-i\theta_{\mathbf{k}}J_y}$$

$$\cos\theta_{\mathbf{k}} = k_z / \sqrt{q^2+k_z^2}$$

$$g_{lm}^{l'm'}(\mathbf{q};z,z_0)=\frac{e^{-i(m-m')\varphi_{\mathbf{q}}}}{\sqrt{\sigma_l\sigma_{l'}}}\left[\mathrm{sgn}(z-z_0)\right]^{l+l'+m+m'}\sum_{m_1=-l}^{l}\sum_{m_2=-l'}^{l'}\sum_{\mu,\lambda_\mu>0}d_{mm_1}^{l}[i\tau(q\lambda_\mu)]$$

$$\times\left\langle lm_1\middle|\psi_\mu\right\rangle\frac{e^{-Q_\mu(q)|z-z_0|}}{\lambda_\mu^2Q_\mu(q)}\left\langle\psi_\mu\middle|l'm_2\right\rangle d_{m'm_2}^{l'}[i\tau(q\lambda_\mu)]$$

$$Q_\mu(q)=\sqrt{q^2+1/\lambda_\mu^2}$$

$$\cos[i\tau(x)]=\sqrt{1+x^2}\ ,\ \ \sin[i\tau(x)]=-ix$$

$$D_{mm'}^{l}(\alpha,\beta,\gamma)=e^{-im\alpha}d_{mm'}^{l}(\beta)e^{-im'\gamma}$$

Evanescent Waves

Plane waves:

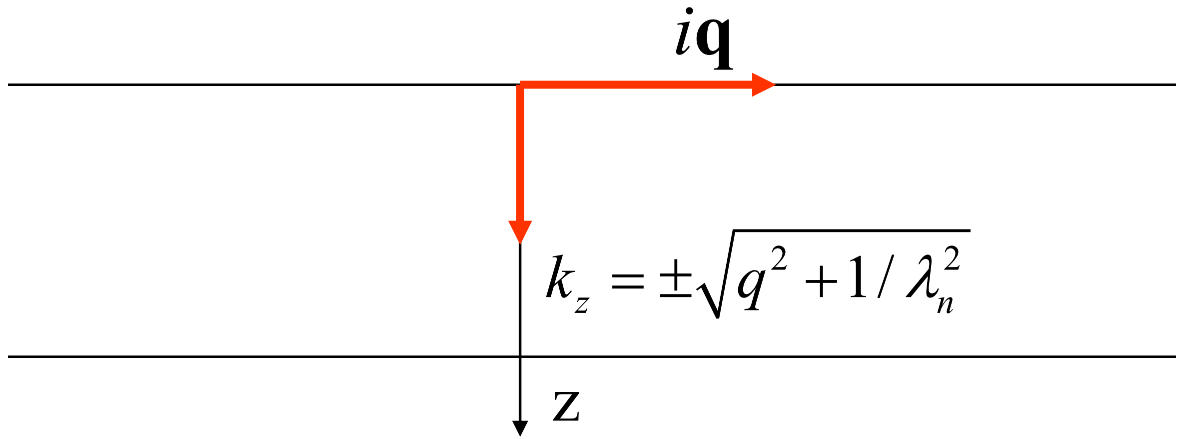
$$I_{\mathbf{k}} = e^{-\mathbf{k}\cdot\mathbf{r}} \; F_{\mathbf{k}}(\hat{\mathbf{s}}), \qquad \mathbf{k}\cdot\mathbf{k} = 1/\lambda_\mu^2, \; \mathbf{k} = k\hat{\mathbf{n}}$$

$\hat{\mathbf{n}}$ - real unit vector

$$F_{\mathbf{k}}(\hat{\mathbf{s}}) = \sum_{lm} \langle lm | \psi_\mu \rangle Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{k}})$$

Evanescent waves:

$$\mathbf{k} = -i\mathbf{q} \pm \hat{\mathbf{z}}\sqrt{q^2 + 1/\lambda_\mu^2}, \quad \mathbf{q}\cdot\hat{\mathbf{z}} = 0, \quad \mathbf{k}\cdot\mathbf{k} = 1/\lambda_\mu^2$$



$$I_{\hat{\mathbf{k}}Mn} = \exp\left(-\frac{\hat{\mathbf{k}}\cdot\mathbf{r}}{\lambda_{Mn}}\right)\sum_{lm} Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{z}})\frac{\exp(-im\varphi_{\mathbf{k}})}{\sqrt{\sigma_l}} d^l_{mM}(\theta_{\mathbf{k}})\langle l|\phi_n(M)\rangle$$

(plane wave modes)

$$I^{(\pm)}_{\mathbf{q}Mn} = \exp\left[i\mathbf{q}\cdot\boldsymbol{\rho} - Q_{Mn}(q)z\right]\sum_{lm} Y_{lm}(\hat{\mathbf{s}};\hat{\mathbf{z}})\frac{\exp(-im\varphi_{\mathbf{q}})}{\sqrt{\sigma_l}}(-1)^{l+m} d^l_{m,-M}(\theta_{\mathbf{k}})\langle l|\phi_n(M)\rangle$$

(evanescent modes)

$$G_0(\mathbf{r},\hat{\mathbf{s}};\mathbf{r}_0,\hat{\mathbf{s}}_0) = \sum_{Mn}\int \frac{d^2q}{(2\pi)^2} I^{(\pm)}_{\mathbf{q}Mn}(\mathbf{r},\hat{\mathbf{s}}) V_{\mathbf{q}Mn} I^{(\mp)}_{-\mathbf{q}Mn}(\mathbf{r}_0,-\hat{\mathbf{s}}_0)$$

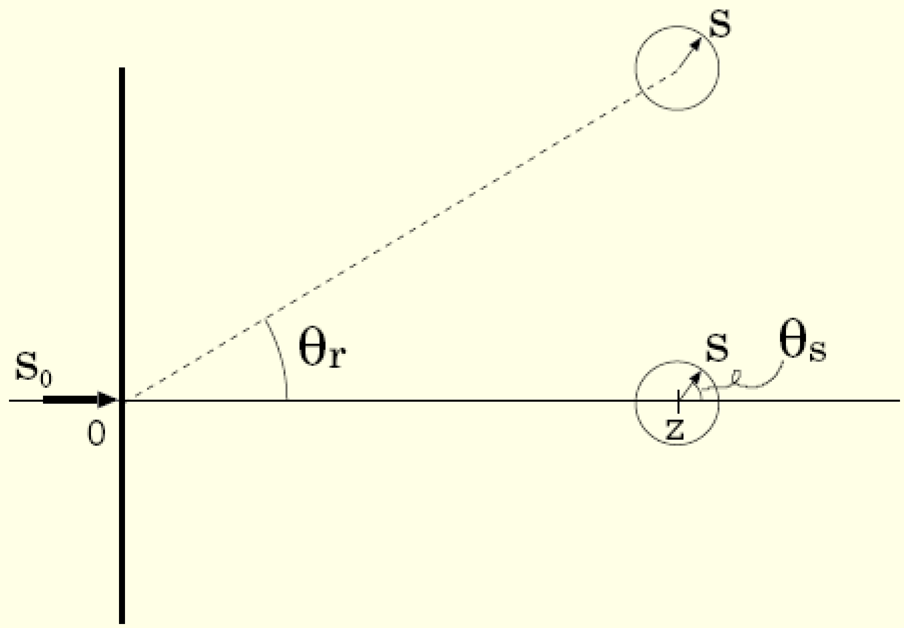
$$V_{\mathbf{q}Mn} = \frac{1}{Q_{Mn}(q)\lambda^2_{Mn}}$$

Half-space problem

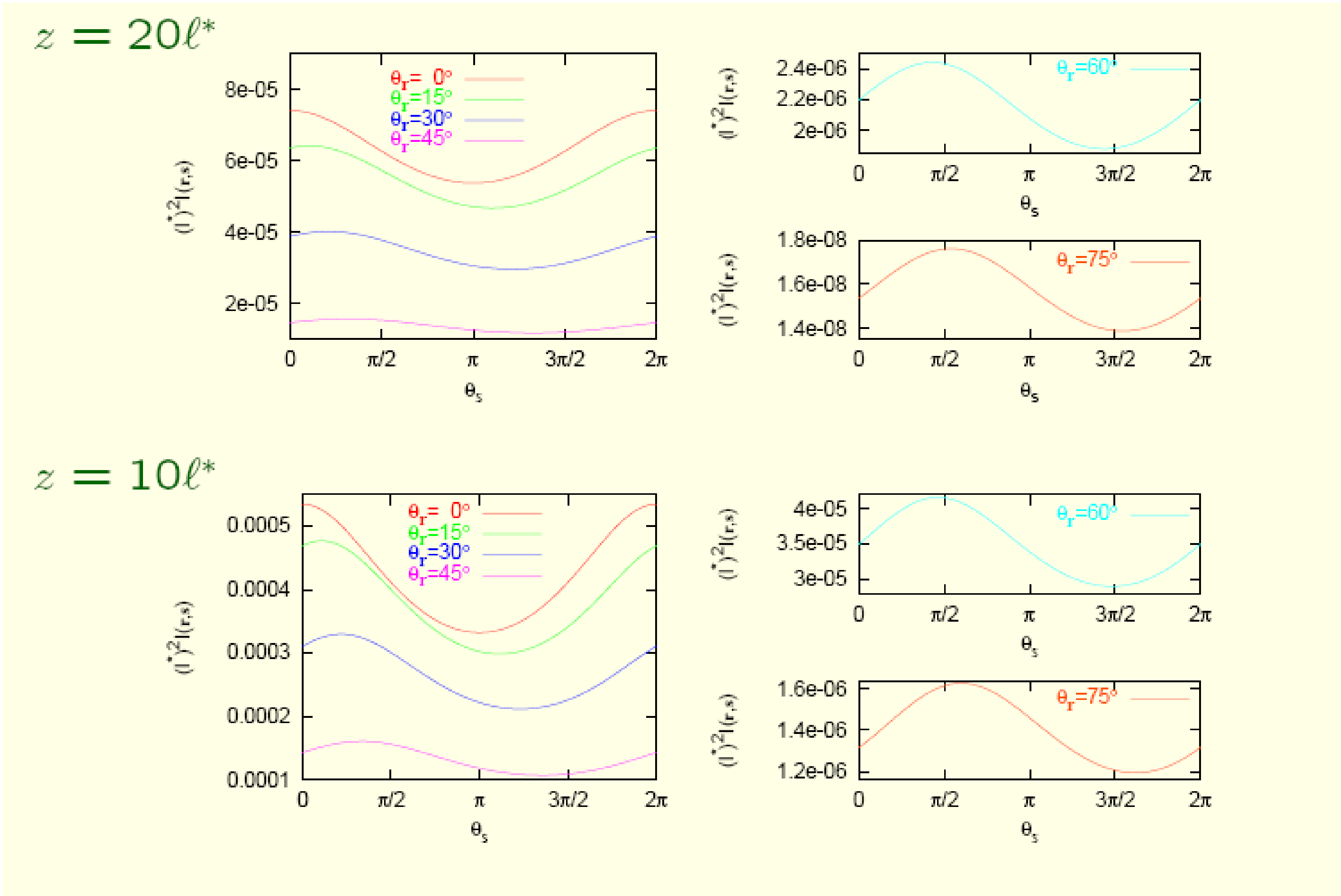
z<0
Nonscattering medium

Z>0
Scattering medium
Propagationd escribed by the RTE

$$I = \int \frac{\mathrm{d}^2 q}{(2\pi)^2} \sum_{Mn} A_{Mn}(\mathbf{q}) I^{(+)}_{\mathbf{q}Mn}$$

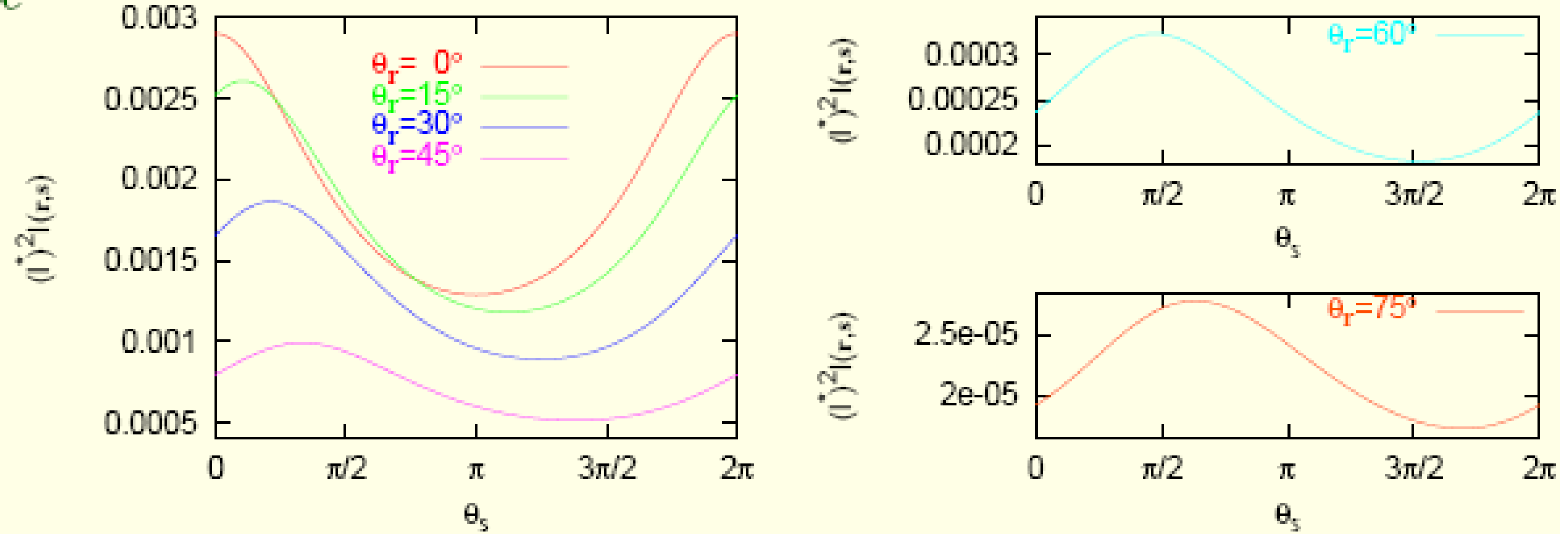


$\mu_a/\mu_s = 6.0 \times 10^{-5}, \quad g = 0.98.$

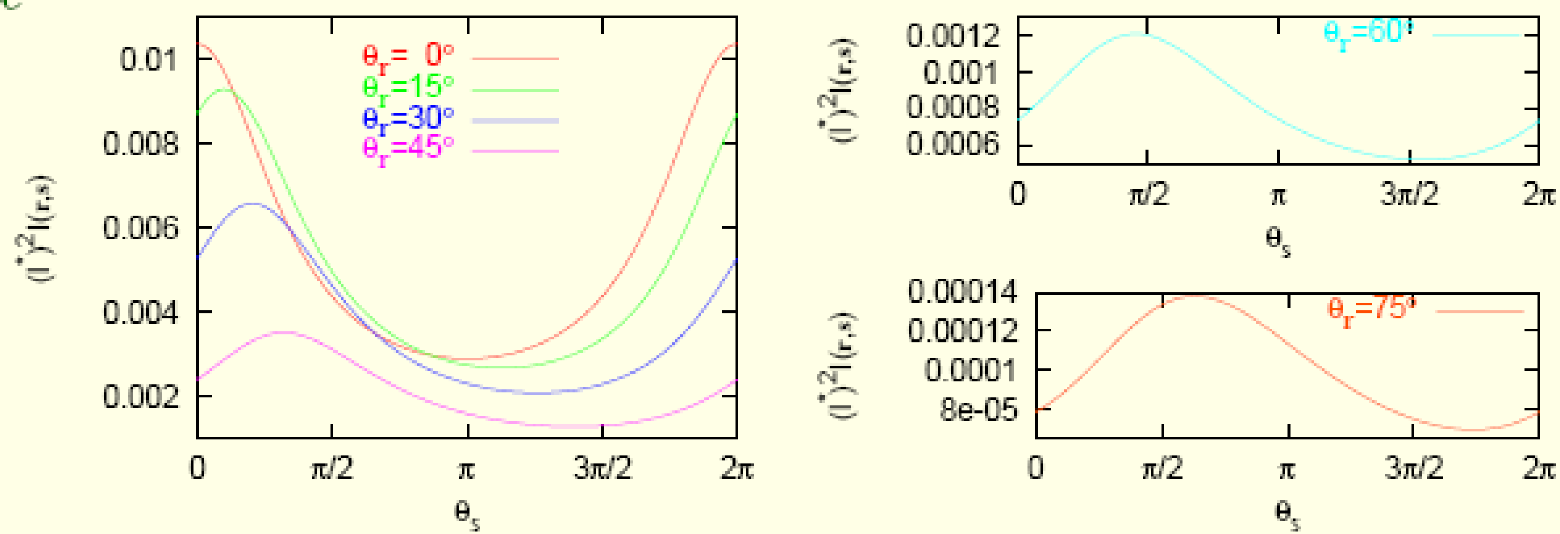


z = 20ℓ*
θr= 0°
θr=15°
θr=30°
θr=45°
8e-05
6e-05
4e-05
2e-05
(l*)²I(r,s)
0
π/2
π
3π/2
2π
θs
θr=60°
2.4e-06
2.2e-06
2e-06
θr=75°
1.8e-08
1.6e-08
1.4e-08
z = 10ℓ*
0.0005
0.0004
0.0003
0.0002
0.0001
4e-05
3.5e-05
3e-05
1.6e-06
1.4e-06
1.2e-06

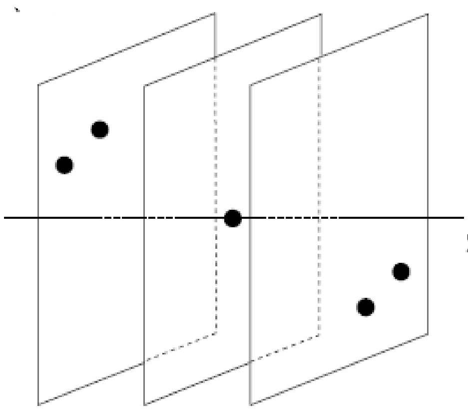
$z = 5\ell^*$


θr= 0°
θr=15°
θr=30°
θr=45°
θr=60°
θr=75°
0.003
0.0025
0.002
0.0015
0.001
0.0005
0.0003
0.00025
0.0002
2.5e-05
2e-05
0
π/2
π
3π/2
2π
θs
(l*)²I(r,s)

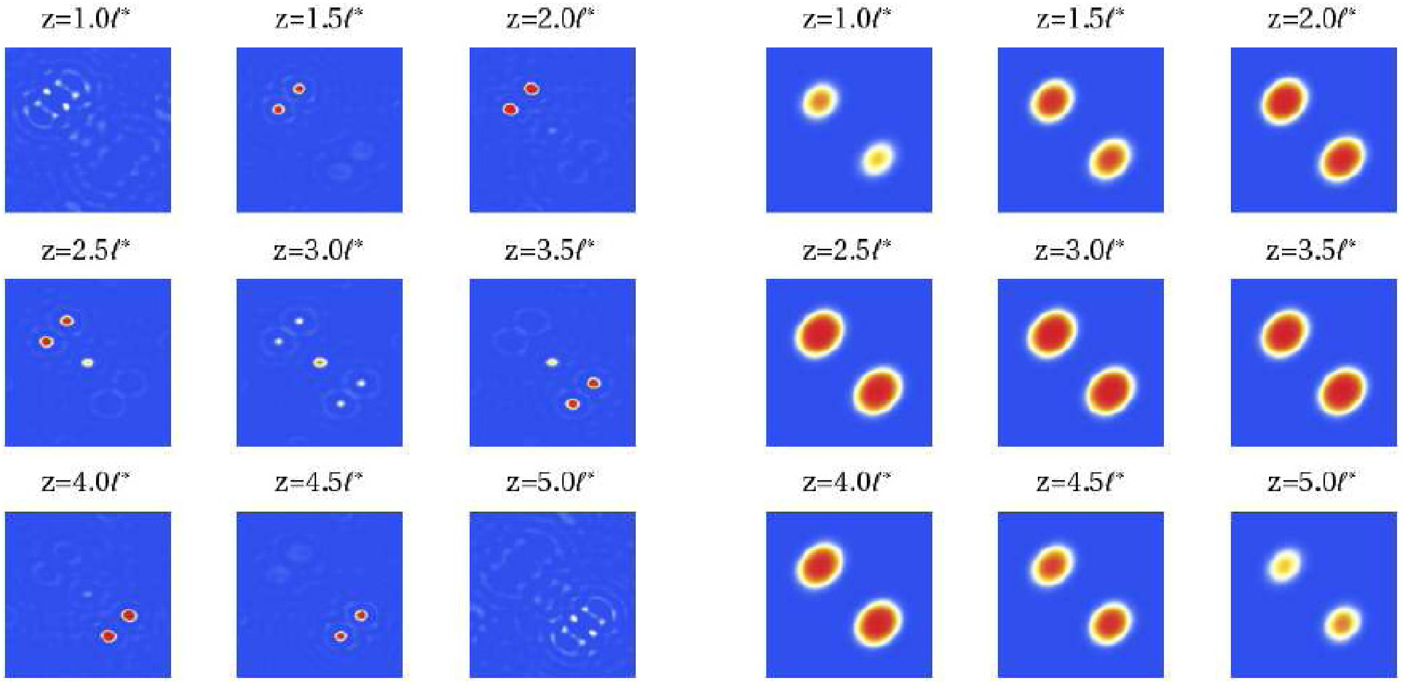
$z = 3\ell^*$


θr= 0°
θr=15°
θr=30°
θr=45°
θr=60°
θr=75°
0.01
0.008
0.006
0.004
0.002
0.0012
0.001
0.0008
0.0006
0.00014
0.00012
0.0001
8e-05
0
π/2
π
3π/2
2π
θs
(l*)²I(r,s)

RESULTS: SIMULATIONS

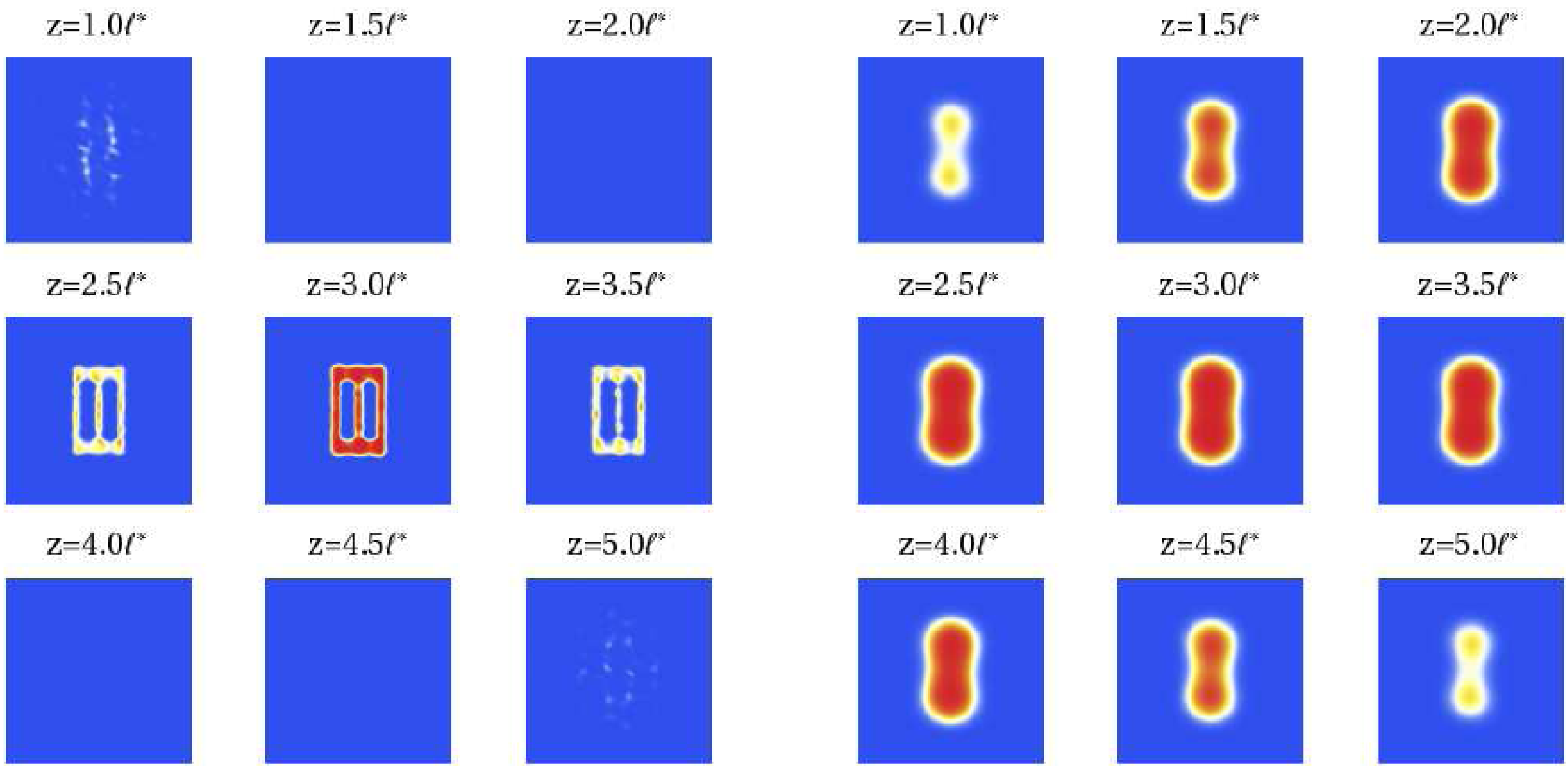


A set of 5 point absorbers
in an L=6ℓ* slab
The field of view is 16ℓ*



RTE Diffusion approximation

A bar target in the center of the same slab

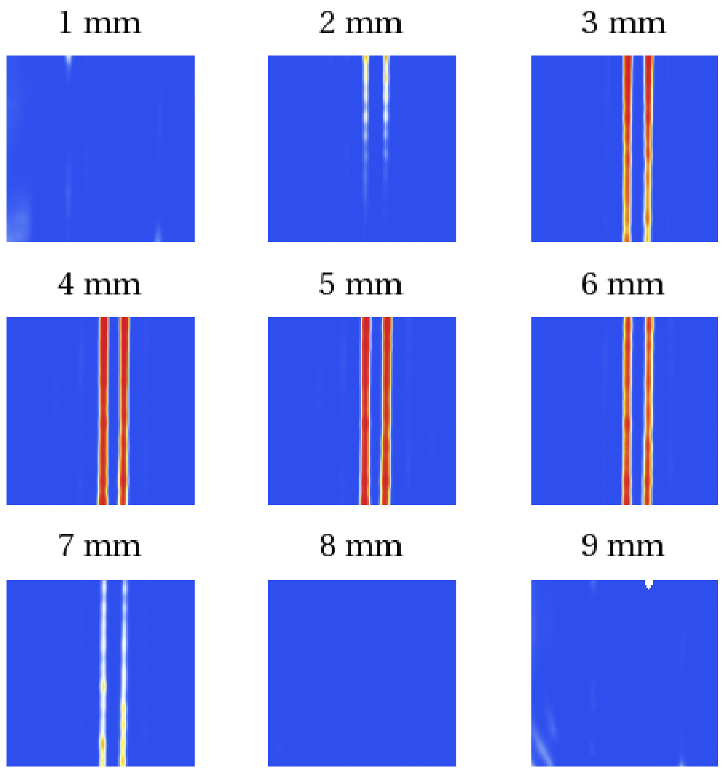


RTE

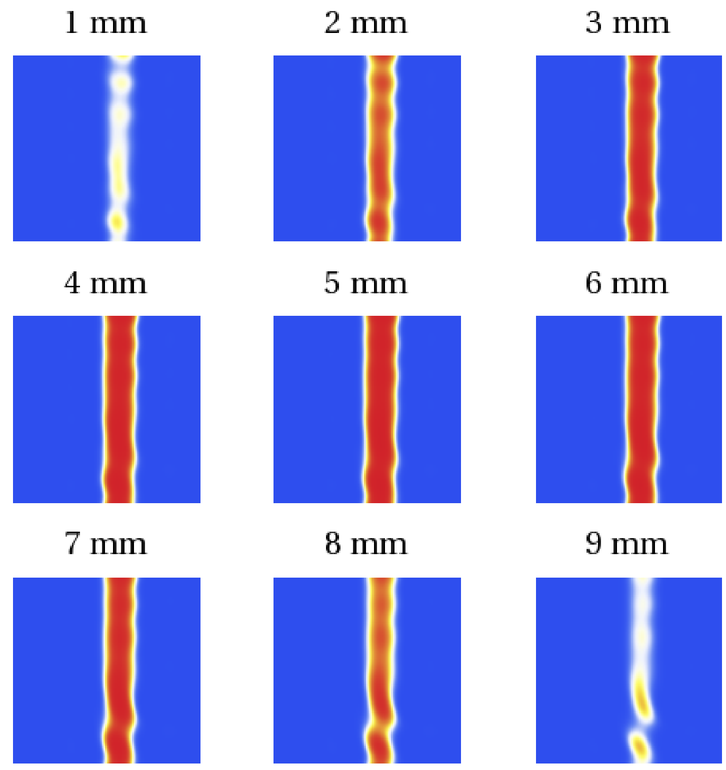
Diffusion approximation

RESULTS: EXPERIMENT

Two thin vertical wires in a 1cm thick slab filled with intralipid solution (looks like milk)



RTE



Diffusion approximation

CONCLUSIONS

- The method of rotated reference frames can be used in optical tomography of mesoscopic samples
- The images are of superior quality compared to those obtained by using the diffusion approximation
- The quality of images can be comparable to that in X-ray tomography because the RTE retains some information about ballistic rays, single-scattered rays, etc.

1. V.A.Markel, “Modified spherical harmonics method for solving the radiative transport equation,” Letter to the Editor, *Waves in Random Media* **14**(1), L13-L19 (2004).

2. G.Y.Panasyuk, J.C.Schotland, and V.A.Markel, “Radiative transport equation in rotated reference frames,” *Journal of Physics A*, **39**(1), 115-137 (2006).

3. J.C.Schotland and V.A.Markel , “Fourier-Laplace structure of the inverse scattering problem for the radiative transport equation,” Inverse Problems and Imaging 1(1), 181-188 (2007).

4. M.Machida, J.C.Schotland and V.A.Markel, “MRRF with boundary conditios,” under consideration in J.Phys.A

5. M.Machida, J.C.Schotland and V.A.Markel, “MRRF with boundary conditios,” under consideration in J.Phys.A

6. M.Machida, J.C.Schotland and V.A.Markel, “Inverse problem with MRRF,” to be submitted to Inv.Prob. in the near future…..