Poynting Vector, Second Law of Thermodynamics and Negative Refraction

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Heating Rate in Material Media (Stationary Case)

$q = \langle \mathbf{J} \cdot \mathbf{E} \rangle_{\text{time}}$ - the heating rate

$\mathbf{J} = \partial \mathbf{P} / \partial t + c\nabla \times \mathbf{M}$ - the total current created by all charges associated with the medium, including conductivity current.

(There are no external or "free" currents)

Heating rate is the "systematic influx of energy (per unit time per unit volume) from external sources of radiation."

LL, VIII

A) Non-Magnetic Medium

$$\mathbf{J} = \partial \mathbf{P} / \partial t$$

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}[\mathbf{E}_\omega(\mathbf{r})e^{-i\omega t}]$$

$$\mathbf{P}(\mathbf{r},t) = \mathrm{Re}[\mathbf{P}_\omega(\mathbf{r})e^{-i\omega t}]$$

$$\mathbf{J}(\mathbf{r},t) = \mathrm{Re}[\mathbf{J}_\omega(\mathbf{r})e^{-i\omega t}]$$

$$\mathbf{P}_\omega(\mathbf{r}) = \frac{\varepsilon_\omega - 1}{4\pi}\mathbf{E}_\omega(\mathbf{r}) \; ; \;\; \mathbf{J}_\omega(\mathbf{r}) = -i\omega \mathbf{P}_\omega(\mathbf{r})$$

$$q = \frac{1}{2}\mathrm{Re}[\mathbf{J}_\omega^* \cdot \mathbf{E}_\omega] = \frac{\omega}{8\pi}\left|\mathbf{E}_\omega\right|^2 \mathrm{Im}(\varepsilon_\omega)$$

The conventional result

B) Magnetic Medium (Volume Term)

(i) $\mathbf{r} \in V \ , \quad \nabla \mu_\omega = 0$

$$\mathbf{J} = \partial \mathbf{P} / \partial t + c \nabla \times \mathbf{M}$$

$$\mathbf{J}_\omega = -i\omega \frac{\varepsilon_\omega - 1}{4\pi} \mathbf{E}_\omega + c \nabla \times \frac{\mu_\omega - 1}{4\pi} \mathbf{H}_\omega$$

$$\mathbf{J}_\omega = -i\omega \frac{\varepsilon_\omega - 1}{4\pi} \mathbf{E}_\omega + c \frac{\mu_\omega - 1}{4\pi} \nabla \times \mathbf{H}_\omega$$

$$\nabla \times \mathbf{H}_\omega = -i \frac{\omega}{c} \mathbf{D}_\omega = -i \frac{\omega}{c} \varepsilon_\omega \mathbf{E}_\omega$$

$$\mathbf{J}_\omega = \frac{\omega}{4\pi i} [\mu_\omega \varepsilon_\omega - 1] \mathbf{E}_\omega$$

$$q^{(V)} = \frac{\omega |\mathbf{E}_\omega|^2}{8\pi} \mathrm{Im}[\mu_\omega \varepsilon_\omega]$$

Not the conventional result

B) Magnetic Medium (Surface Term)

(ii) $\mathbf{r} \in S = \partial V \;,\quad \nabla \mu_\omega(\mathbf{r}) = (\mu_\omega - 1)\hat{\mathbf{n}}_\mathbf{R}\,\delta\big(\hat{\mathbf{n}}_\mathbf{R} \cdot (\mathbf{r} - \mathbf{R})\big)$

R - point on the surface

$\hat{\mathbf{n}}_\mathbf{R}$ - outward unit normal drawn at point **R**

$$q^{(S)}(\mathbf{R}) = \frac{\omega}{8\pi} \mathrm{Re}\Big[(1 - \mu_\omega)\big(\mathbf{H}_\omega \times \mathbf{E}^*_\omega\big) \cdot \hat{\mathbf{n}}_\mathbf{R}\Big]$$

$$Q = \underbrace{\int q^{(V)}(\mathbf{r})\, d^3 r}_{Q^{(V)}} + \underbrace{\int q^{(S)}(\mathbf{R})\, d^2 R}_{Q^{(S)}}$$

$$= Q^{(V)} + Q^{(S)}$$

(total heat absorbed by the body per unit time)

Intermediate Summary

The conventional result

$$q_{\text{conv}} = \frac{\omega}{8\pi}\left[\left|\mathbf{E}_{\omega}\right|^{2}\varepsilon''_{\omega} + \left|\mathbf{H}_{\omega}\right|^{2}\mu''_{\omega}\right]$$

$$Q_{\text{conv}} = \int q_{\text{conv}}(\mathbf{r})\,d^{3}r \qquad \text{(volume term only)}$$

My result

$$q_{\text{my}}^{(V)} = \frac{\omega\left|\mathbf{E}_{\omega}\right|^{2}}{8\pi}\,\text{Im}\left[\mu_{\omega}\varepsilon_{\omega}\right]$$

$$q_{\text{my}}^{(S)}(\mathbf{R}) = \frac{\omega}{8\pi}\,\text{Re}\left[(1-\mu_{\omega})\left(\mathbf{H}_{\omega}\times\mathbf{E}_{\omega}^{*}\right)\cdot\hat{\mathbf{n}}_{\mathbf{R}}\right]$$

$$Q_{\text{my}} = \int q_{\text{my}}^{(V)}(\mathbf{r})\,d^{3}r + \int q_{\text{my}}^{(S)}(\mathbf{R})\,d^{2}R$$

It can be shown that

$$Q_{\text{my}} = Q_{\text{conv}}$$

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but

$$q_{\text{my}} \neq q_{\text{conv}}$$

Why the Two Results are Different? (Poynting Theorem)

$$\frac{c}{4\pi}\nabla\cdot(\mathbf{E}\times\mathbf{H})+\frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)=0$$

This is an identity which follows from Maxwell equations

If we identify $\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$ **(this is simply postulated)**

Then, in the statioary case (i.e., for monochromatic fields),

$\left\langle\nabla\cdot\mathbf{S}\right\rangle_{\text{time}}+q=0$ ... and

$$q=\frac{1}{4\pi}\left\langle\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right\rangle_{\text{time}}=q_{\text{conv}}$$

The Poynting Theorem (Cont.)

But $\left\langle \frac{1}{4\pi}\left(\mathbf{E}\cdot\frac{\partial \mathbf{D}}{\partial t}+\mathbf{H}\cdot\frac{\partial \mathbf{B}}{\partial t}\right)\right\rangle \neq \langle \mathbf{J}\cdot\mathbf{E}\rangle$!!!

(the equality holds in non-magnetic media only).

There is no physical basis to identify $\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$ as the current of energy (which, by definition, is the Poynting vector).

The formula $\mathbf{S}=\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}$ is incorrect

The Poynting Theorem (Cont.)

Then what is the correct expression for the Poynting vector?

a)

$$\mathbf{S} = \frac{c}{4\pi}\left\langle \mathbf{e} \times \mathbf{h} \right\rangle_{\text{volume}} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} + \text{terms quadratic in fluctuations}$$

b)

If we take $\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B}$, then

$$\left\langle \nabla \cdot \mathbf{S} \right\rangle_{\text{time}} + \left\langle \mathbf{J} \cdot \mathbf{E} \right\rangle_{\text{time}} = 0$$

and $q = \left\langle \mathbf{J} \cdot \mathbf{E} \right\rangle_{\text{time}} = q_{\text{my}}$

Heating Rate and Negative Refraction

$$q_{\mathrm{my}}^{(V)} = \frac{\omega \left| \mathbf{E}_{\omega} \right|^{2}}{8\pi} \mathrm{Im}\left[\mu_{\omega} \varepsilon_{\omega} \right]$$

$$q_{\mathrm{my}}^{(S)}\left(\mathbf{R}\right) = \frac{\omega}{8\pi} \mathrm{Re}\left[\left(1 - \mu_{\omega}\right)\left(\mathbf{H}_{\omega} \times \mathbf{E}_{\omega}^{*}\right) \cdot \hat{\mathbf{n}}_{\mathbf{R}} \right]$$

$$Q_{\mathrm{my}} = Q_{\mathrm{my}}^{(V)} + Q_{\mathrm{my}}^{(S)} = Q_{\mathrm{conv}} > 0$$

Heating rate and Negative Refraction (Cont.)

Negative refraction $\Leftrightarrow \operatorname{Im}[\mu_\omega \varepsilon_\omega] < 0$

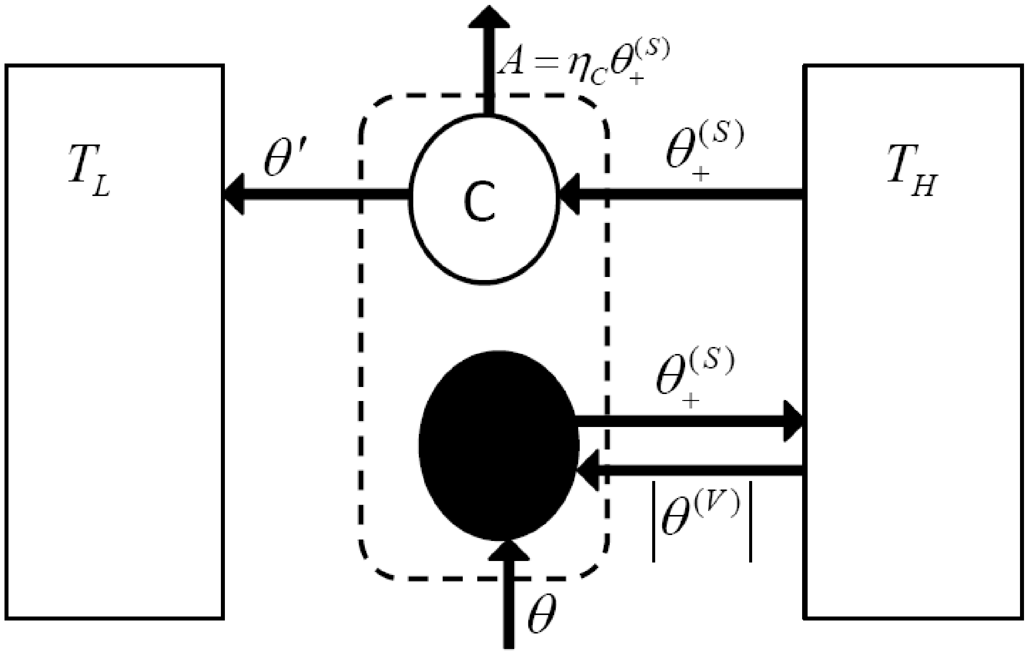
$$k^2 = \mu_\omega \varepsilon_\omega \left(\frac{\omega}{c}\right)^2 \Leftrightarrow \operatorname{Im} k^2 = 2k'k'' \propto \operatorname{Im}[\mu_\omega \varepsilon_\omega]$$

In a medium with negative refraction:

$$Q^{(V)} < 0; \qquad Q^{(S)} = Q + \left|Q^{(V)}\right| > Q$$

Volume is cooled but surface is heated; the heat generated at the surface is larger that the total absorbed energy. Is this possible?

Thermodynamic Considerations



$$\theta = Q\Delta t$$

$$\theta = \theta_+^{(S)} - \left|\theta^{(V)}\right| > 0$$

$$\eta = \frac{A}{\theta} = \eta_C \frac{1}{1 - \left|\theta^{(V)}\right| / \theta_+^{(S)}} > \eta_C$$

If $\frac{T_L}{T_H} < \frac{\left|\theta^{(V)}\right|}{\theta_+^{(S)}}$, then $\eta > 1$

Conclusions

- The correct expression for the Poynting vector involves **B** instead of **H.**
- Consequently, the direction of phase velocity and the PV always coincide.
- If a media exhibits negative refraction, radiation cools its volume and heats its surface. This is prohibited by the second law.
- Negative refraction is impossible.
- The results hold for active or passive media and for media with spatial dispersion.
- But the results do not hold in photonic crystals where the group and phase velocities can, in principle, be anti-parallel. However, effective medium parameters are inapplicable to such structures [e.g., see A.A.Govyadinov, V.A.Markel, Phys. Rev. B **78**, 035403 (2008)]

Details at arXiv:0712.0605

Objections

1. "You have assumed that only electric field does work on moving charges. But magnetic field can also do work, e.g., on magnetic moments. Otherwise, how do you explain ...." [choose your example]

Dead men tell no tales and magnetic field does no work

Objections (Cont.)

2. "May be, effective medium parameters describe (approximately) some phenomena associated with wave propagation through composite media but not *all* such phenomena."

The experimentally measurable quantities, such as the intensity, are bilinear in the fields. Therefore, any useful homogenization model must correctly predict such quadratic combinations, including the Poynting vector and the heating rate.

Objections (Cont.)

3. "The electric and magnetic currents are different in physical origin and the same laws of motion can not be applied to them."

There is neither physical nor mathematical way to disentangle the terms $\partial \mathbf{P} / \partial t$ and $c\nabla \times \mathbf{M}$ from the expression for the total current

$$\mathbf{J} = \partial \mathbf{P} / \partial t + c\nabla \times \mathbf{M}$$

All laws or formulas must be applied to the total current **J**.

Objections (Cont.)

4. “What about the zero-frequency limit? Won’t you obtain unphysical results at the surface?”

First, we are not really talking about low frequency magnetism.

But there is no unphysical effects in my theory in the zero-frequency limit. In fact, a careful analysis reveals that the conventional theory predicts an electromagnetic equilibrium for a magnetized object in external static E-field which contradicts mechanical equilibrium of charges. My theory is free of such contradictions.

Objections (Cont.)

5. “What about all those experiments ?”

Show me your superselns.

(and it’s been in the works for about ten years)