Nonlinear corrections for inversion formulas in optical diffusion tomography

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Abstract: We propose a new image reconstruction algorithm for solving the nonlinear inverse problem of optical diffusion tomography. The algorithm is based on the analytic construction of the pseudo-inverse operator for the linearized problem which is then used to obtain nonlinear corrections to arbitrary order. The method is illustrated with numerical simulations.

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There has been considerable interest in the inverse scattering problem for diffuse light [1,2]. The basic physical problem consists of reconstructing the spatial distribution of the optical absorption and diffusion coefficients inside a highly-scattering medium from intensity measurements on the boundary of the medium.

The equations describing scattering of diffuse light from fluctuations in the absorption and diffusion coefficients α and D are, in general, nonlinear [2,3]. Thus numerical methods for solving the nonlinear inverse problem have been widely studied and are typically based on Newton's method [4]. A limitation of this approach is due to its computational complexity which arises from the fact that the forward problem must be solved at each iteration of the algorithm.

Approaches to the inverse problem based on linearization of the forward problem have also been proposed. In this method, the integral equations of diffuse light propagation are expanded and linearized in α and D [5,6]. These equations can then be solved with the use of analytic inversion formulas [7–9]. The use of inversion formulas is especially attractive due to its computational efficiency.

In this work we obtain a solution to the nonlinear problem starting from the linearized solution as a first approximation. This novel approach leads directly to a series expansion which is similar in form to perturbation expansions for the forward problem. It is important to emphasize that our approach is not iterative and does not a solution to the forward problem at for each correction.

In diffusion tomography, measurements are usually taken with point-like source-detector pairs which are placed on a measurement surface which surrounds the medium which is to be imaged. It can be shown that results of such measurements are given by the Green's function G for the diffusion equation. The scattering data can be defined as $\Phi = G_0 - G$ where G_0 is the Green's function in a homogeneous reference medium. Our mathematical approach is based on the observation that the perturbative expansion of G can be inverted order by order. Consider the Dyson equation for G

$$G = G_0 - G_0 V G \ . \tag{1}$$

Here $V = \delta\alpha(\mathbf{r}) - \nabla \cdot D(\mathbf{r})\nabla$ is the appropriately defined perturbation, where $\delta\alpha(\mathbf{r})$ and $\delta D(\mathbf{r})$ are the deviations of the absorption and diffusion coefficients from their average background values. The series expansion for G is obtained by iteration of (1)

$$G = G_0 \sum_{k=0}^{\infty} (-VG_0)^k = \sum_{k=0}^{\infty} (-G_0 V)^k G_0 \ . \tag{2}$$

Now we use expansion (2) and write

$$\Phi = G_0 V G_0 \sum_{k=0}^{\infty} (-VG_0)^k \ . \tag{3}$$

The relation between the scattering data and the unknown functions $\delta\alpha$ and δD can be written in the following general form

$$|\phi\rangle = K_1|\psi\rangle - K_2|\psi\rangle|\psi\rangle + K_3|\psi\rangle|\psi\rangle|\psi\rangle - \dots \ , \tag{4}$$

where we have introduced the state vectors $|\psi\rangle$ and $|\phi\rangle$ which are given in coordinate representation by

$$\psi(\mathbf{r}) = \langle \mathbf{r}|\psi\rangle = \begin{pmatrix} \psi_1(\mathbf{r}) \\ \psi_2(\mathbf{r}) \end{pmatrix} = \begin{pmatrix} \delta\alpha(\mathbf{r}) \\ \delta D(\mathbf{r}) \end{pmatrix} \ , \quad \phi(\mathbf{r}_s, \mathbf{r}_d) = \langle \mathbf{r}_s, \mathbf{r}_d|\phi\rangle = \langle \mathbf{r}_s|\Phi|\mathbf{r}_d\rangle \ . \tag{5}$$

Here $\mathbf{r}_s$ and $\mathbf{r}_d$ are the source and detector coordinates which are limited to the measurement surface, while the variable $\mathbf{r}$ is inside the medium. Note that $|\phi\rangle$ and $|\psi\rangle$ in general belong to different Hilbert spaces.

The tensor operators K_n depend on the geometry of the problem and the type of boundary conditions imposed on the measurement surface. In the linearized case, only the first term in the right hand side of (4) is present. Then the problem is solved by finding the pseudo-inverse of K_1, denoted K_1^+. We have recently obtained this pseudo-inverse of the linearized operator in a number of geometries [8–10]. Assuming that all forward operators K_n as well as the pseudo-inverse K_1^+ are known, we act with K_1^+ on both sides of (4) and use $K_1^+ K_1 = 1$ to obtain

$$|\psi\rangle = K_1^+ \left(|\phi\rangle + K_2|\psi\rangle|\psi\rangle - K_3|\psi\rangle|\psi\rangle|\psi\rangle + \dots\right) \ . \tag{6}$$

Using this result we can express $|\psi\rangle$ as the infinite series

$$|\psi\rangle = |\psi^{(1)}\rangle + |\psi^{(2)}\rangle + |\psi^{(3)}\rangle + \dots \ . \tag{7}$$

The first-order term $|\psi^{(1)}\rangle$ is given by the linearized solution $|\psi^{(1)}\rangle = K_1^+|\phi\rangle$ and the next few orders are as of the form

$$|\psi^{(2)}\rangle = K_1^+ K_2 |\psi^{(1)}\rangle|\psi^{(1)}\rangle \ , \tag{8}$$

$$|\psi^{(3)}\rangle = K_1^+ \left[K_2 \left(|\psi^{(1)}\rangle|\psi^{(2)}\rangle + |\psi^{(2)}\rangle|\psi^{(1)}\rangle\right) - K_3|\psi^{(1)}\rangle|\psi^{(1)}\rangle|\psi^{(1)}\rangle\right] \ . \tag{9}$$

We have performed numerical simulations and obtained the linearized solution and the first nonlinear correction to it. We used the slab geometry in which sources are located on the plane $z = 0$ and detectors on the plane $z = L$ with free boundary conditions. The forward data were calculated for a spherical inhomogeneity of radius $R = 0.4L$. The inhomogeneity was characterized by a different diffuse wavenumber. For continuous-wave measurements, the diffuse wave number is defined by $k^2(\mathbf{r}) = \alpha(\mathbf{r})/D(\mathbf{r})$. We assumed that outside the inhomogeneity $k = k_1 = 2\pi/L$ and inside $k = k_2 > k_1$. Further, we assumed that the diffusion coefficient is constant $D = D_0$ and changes in k are entirely due to changes in the absorption coefficient. The forward data were calculated from the exact solution to the diffusion equation for the spherical inhomogeneity.

The result of calculating the first nonlinear correction to $\delta\alpha$ is illustrated in Fig.1. In this figure we show the reconstruction of $\delta\alpha$ along a line which is parallel to the measurement plane and crosses the spherical inhomogeneity at different depths z. The true profile of $\delta\alpha$ is obviously a step function (shown by short dash). The linearized solution is shown by a long dash. It can be seen that it becomes increasingly less accurate when the mismatch of wave numbers inside and outside the inhomogeneity increases. The deviation of the linearized solution from the true profile is manifested by the appearance of a false void in the center of the sphere. Note that if the mismatch is increased further, the linearized formula results in a reconstruction of only a thin spherical shell which coincides with the true surface of the inhomogeneity (data not shown). This is explained by the fact that for an inhomogeneity with large k_2 the diffuse waves do not enter the internal regions due to strong absorption — an effect that is not accounted for to lowest order in perturbation theory.

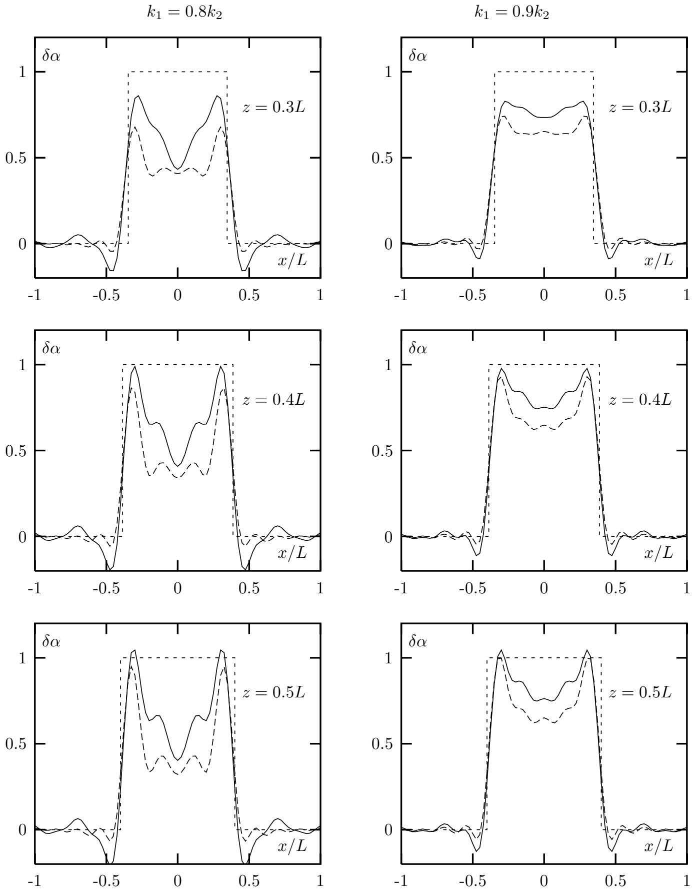

Fig.1. Reconstructed function $\delta\alpha(\mathbf{r})$ calculated along the line $z =$ const (as indicated in the legend), $y = 0$, $x \in [-L, L]$. Long dash: $\delta\alpha = \delta\alpha^{(1)}$ (linearized inversion); solid line: $\delta\alpha = \delta\alpha^{(1)} + \delta\alpha^{(2)}$ (first nonlinear correction); short dash: the true profile of $\delta\alpha$.

To conclude, we have demonstrated that the first nonlinear correction provides better reconstructed images. As can be seen from comparing the panels for $k_1 = 0.8k_2$ and $k_1 = 0.9k_2$, the effect of the first nonlinear correction is to fill the false "voids" that are seen in the linearized inversion. As one could expect, the first nonlinear correction made no significant effect in the cases $k_1 = 0.99k_2$ and $k_1 = 0.5k_2$ (data not shown). In the first case, the linearized inversion already provides accurate results and all higher-order corrections are small. In the second case, the first Born approximation is strongly violated for the forward problem, and higher-order corrections must be included to obtain convergence (provided, the series converges at all). We will address the possibility of numerical calculation of higher-order corrections in future work.

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