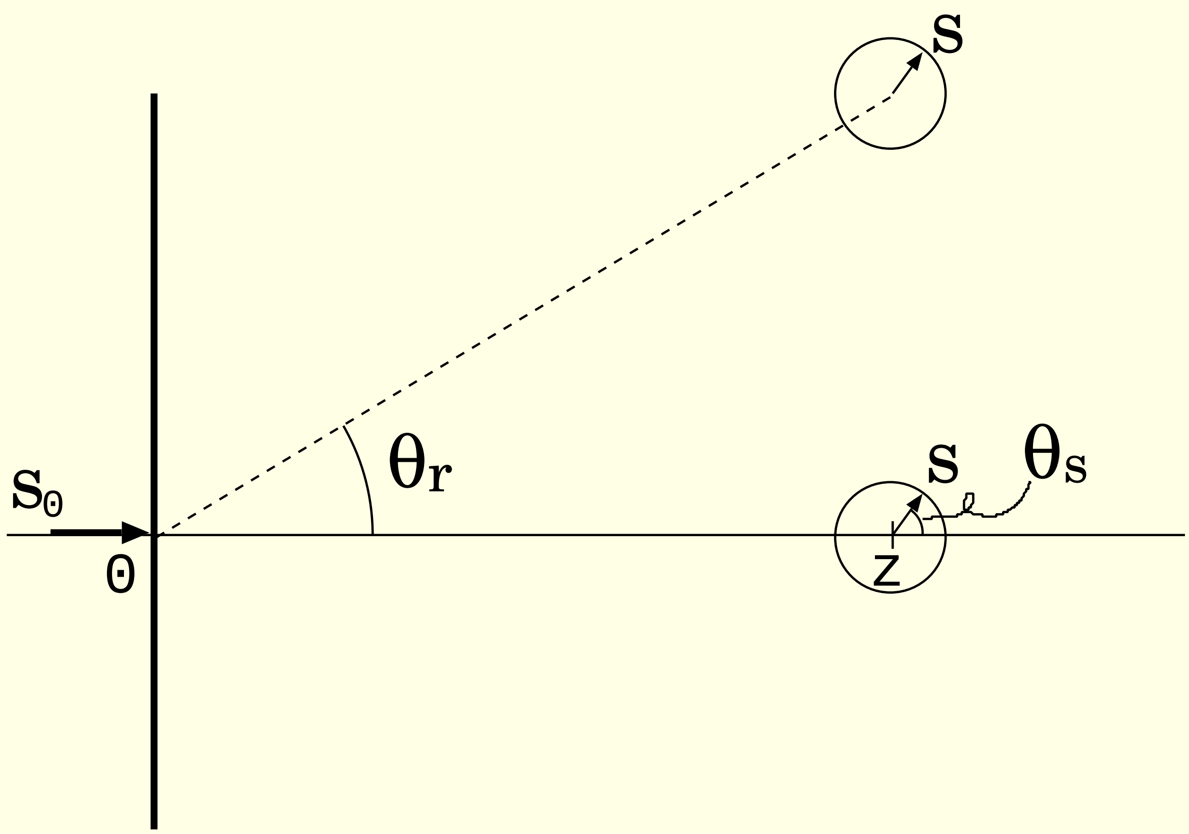
Method of Rotated Reference Frames

For the Half Space

Manabu Machida

1 Half-Space Geometry



$$\mu_a/\mu_s = 6.0 \times 10^{-5}, \quad g = 0.98.$$

2 RTE in the Half Space

RTE:

$$\left[\hat{\mathbf{s}} \cdot \nabla + \mu_{\mathsf{t}} - \mu_{\mathsf{s}} \hat{A}\right] I(\mathbf{r}, \hat{\mathbf{s}}) = \varepsilon(\mathbf{r}, \hat{\mathbf{s}}),$$

where $\mu_{\mathsf{t}} = \mu_{\mathsf{a}} + \mu_{\mathsf{s}}$ and $\hat{A} I(\mathbf{r}, \hat{\mathbf{s}}) = \int \mathrm{d}^2 s' \, A(\hat{\mathbf{s}} \cdot \hat{\mathbf{s}}') I(\mathbf{r}, \hat{\mathbf{s}}')$.

Boundary Condition:

$$I(\boldsymbol{\rho}, z = 0, \hat{\mathbf{s}}) = \delta\left(\boldsymbol{\rho} - \boldsymbol{\rho}_0\right) \delta\left(\hat{\mathbf{s}} - \hat{\mathbf{s}}_0\right) \quad \left[\hat{\mathbf{s}} \cdot \hat{z} > 0\right],$$

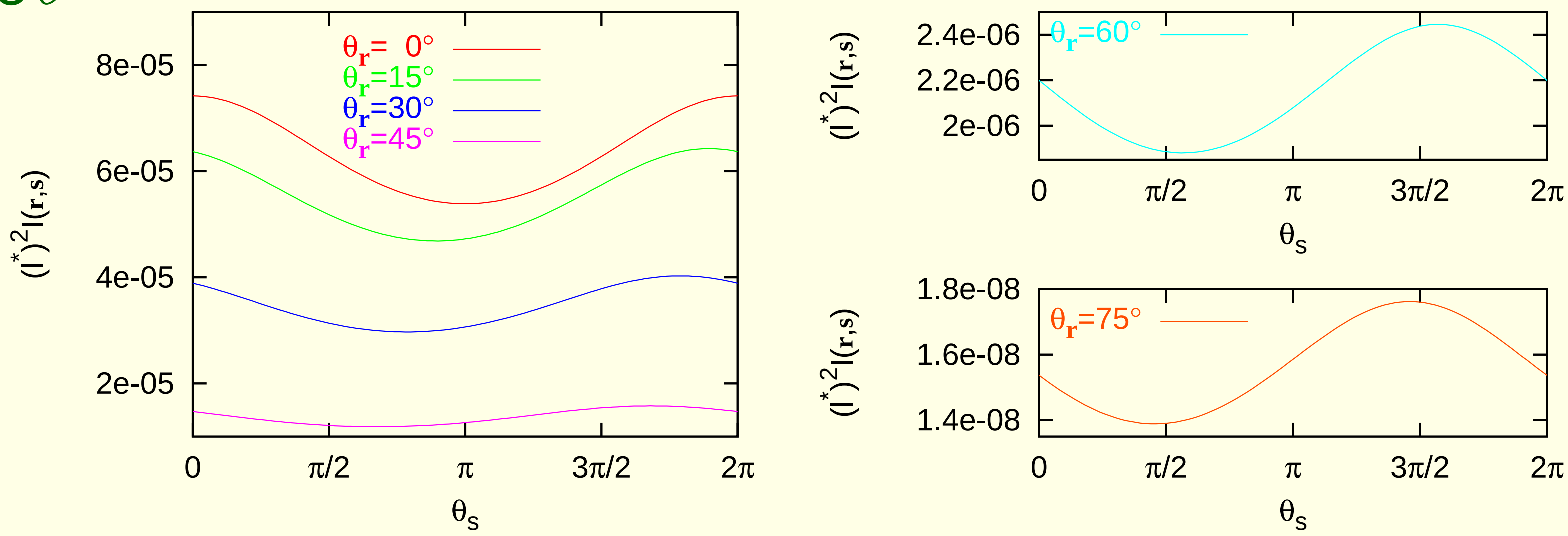
where $\boldsymbol{\rho}_0 = \mathbf{0}$, $\hat{\mathbf{s}}_0 = \hat{z}$.

The Solution:

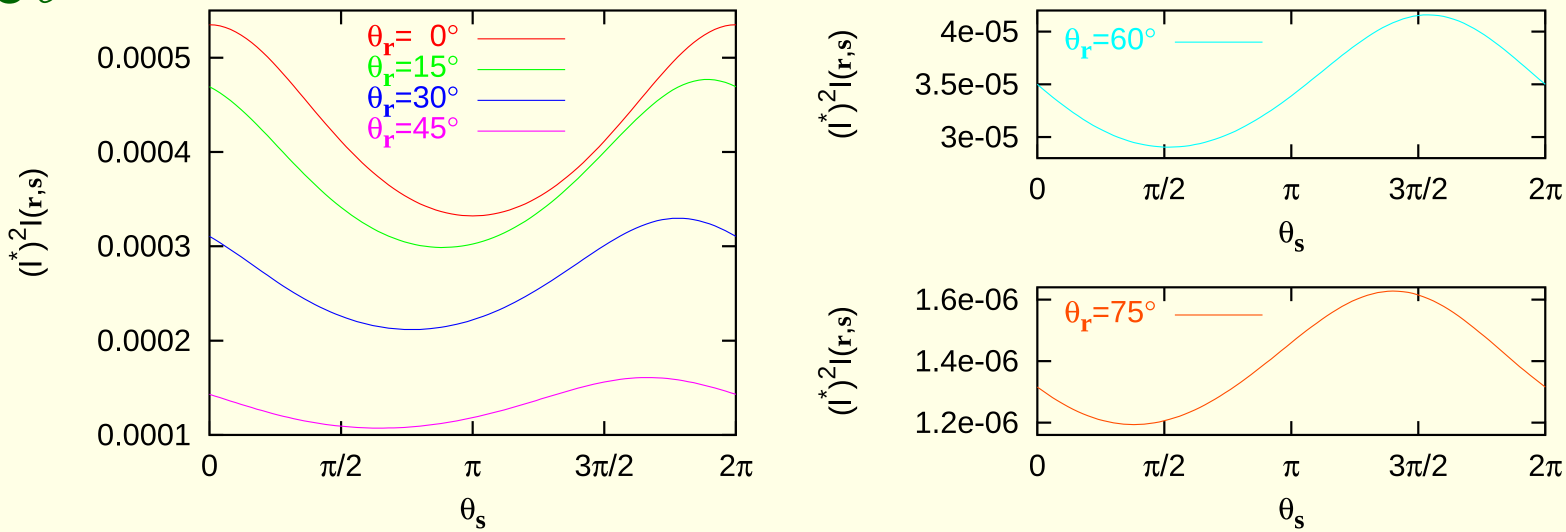
$$I(\mathbf{r}, \hat{\mathbf{s}}) = \sum_{\mu=(M,n)}{}' \int \frac{\mathrm{d}^2 q}{(2\pi)^2} F_\mu(\mathbf{q}) I^{(+)}_{\mathbf{q}\mu}(\mathbf{r}, \hat{\mathbf{s}}).$$

3 Angular Dependence of Intensity 1

$z = 20\ell^*$

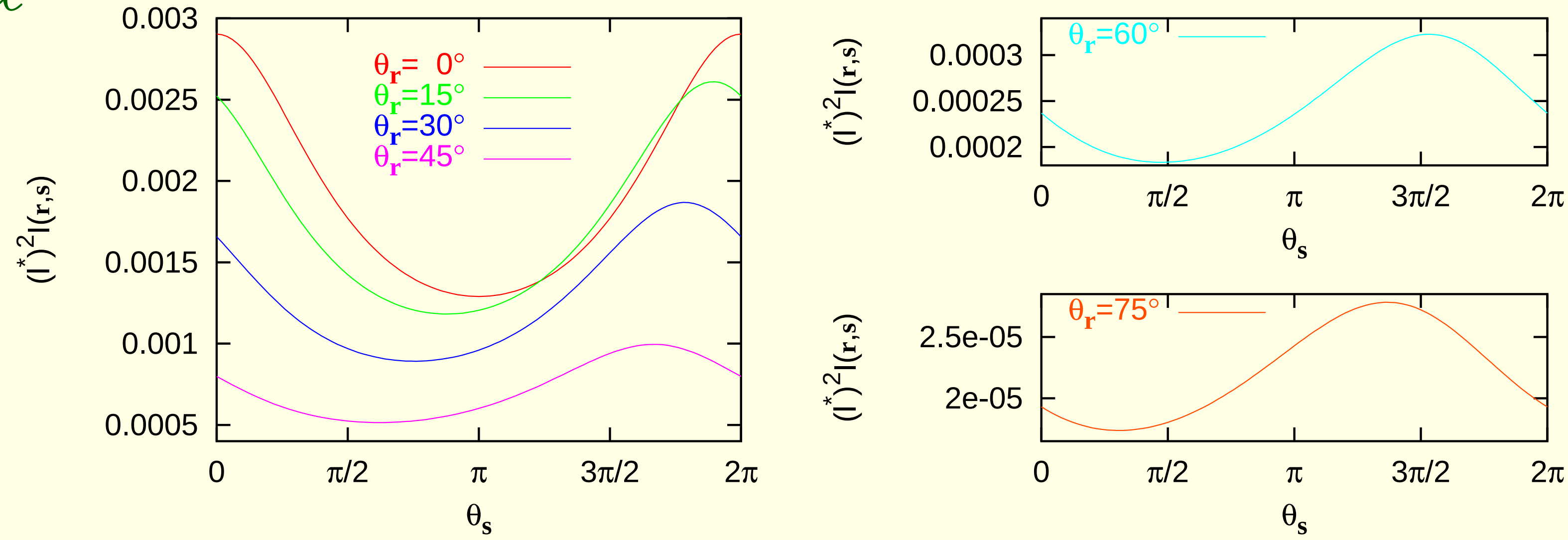


$z = 10\ell^*$

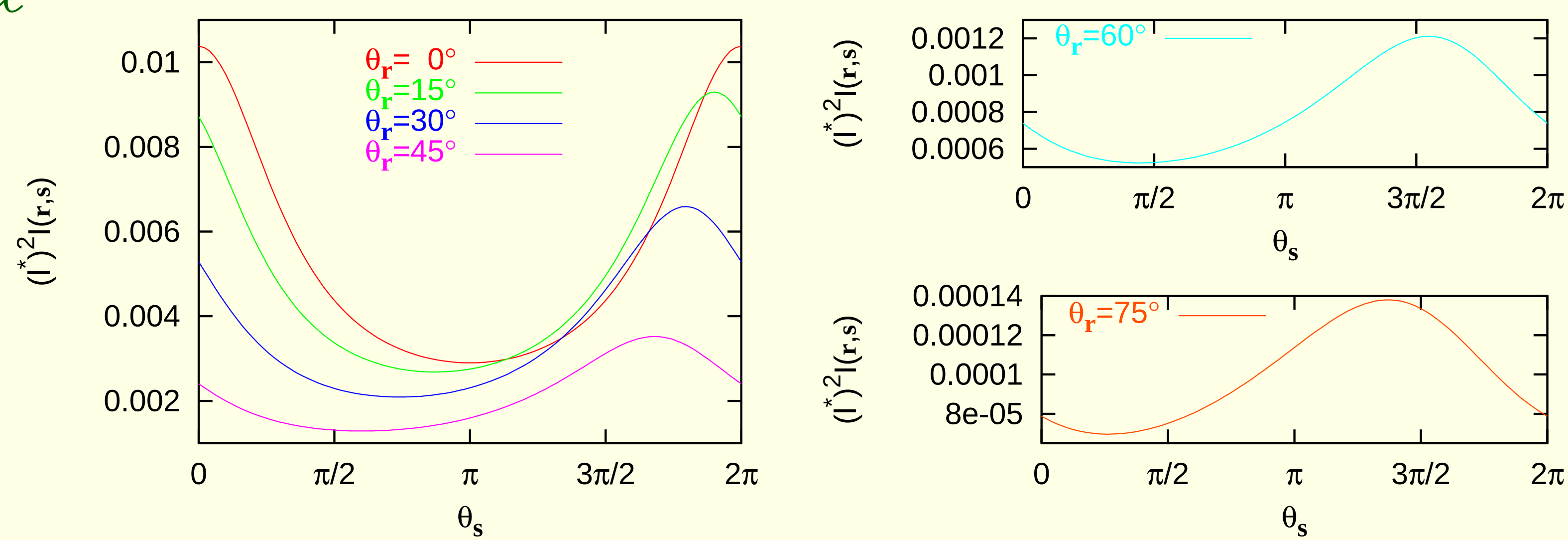


4 Angular Dependence of Intensity 2

$z = 5\ell^*$



$z = 3\ell^*$



5 Basis Modes

Find the solution to

$$\left[\widehat{\mathbf{s}} \cdot \nabla + \mu_{\mathrm{t}} - \mu_{\mathrm{s}} \widehat{A}\right] I_{\mathbf{k}}(\mathbf{r}, \widehat{\mathbf{s}}) = 0$$

with $I_{\mathbf{k}}(\boldsymbol{\rho}, z \to \infty, \widehat{\mathbf{s}}) = 0$

Answer:

$$\mathbf{k} = -i\mathbf{q} + Q_\mu(q)\widehat{z}, \quad Q_\mu(q) = \sqrt{q^2 + \lambda_\mu^{-2}}.$$

$$\begin{aligned} I_{\mathbf{k}}(\mathbf{r}, \widehat{\mathbf{s}}) &= I_{\mathbf{q}\mu}^{(+)}(\mathbf{r}, \widehat{\mathbf{s}}) \\ &= \mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{\rho} - Q_\mu(q) z} \sum_{\ell=|M|}^{\ell_{\max}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\widehat{\mathbf{s}}) \frac{\mathrm{e}^{-im\phi_{\mathbf{q}}}}{\sqrt{\sigma_\ell}} \\ &\times \langle \ell | \phi_n(M) \rangle \, d^{\ell}_{mM}[i\tau(q\lambda_\mu)]. \end{aligned}$$

6 Eigenvalues and Eigenvectors

$$B(M)\,|\phi_n(M)\rangle = \lambda_{\mu=(M,n)}\,|\phi_n(M)\rangle\,.$$

Here

$$B(M) = \begin{pmatrix} 0 & \beta_{|M|+1} & 0 & \\ \beta_{|M|+1} & 0 & \beta_{|M|+2} & \\ 0 & \beta_{|M|+2} & 0 & \ddots \\ & & \ddots & \ddots \end{pmatrix},$$

where

$$\beta_\ell = \sqrt{\frac{\ell^2 - M^2}{(4\ell^2-1)\sigma_\ell\sigma_{\ell-1}}}.$$

Note that

$$\sigma_\ell = \mu_{\mathrm{a}} + \mu_{\mathrm{s}}\left(1-g^\ell\right), \quad g = \int \mathrm{d}^2 s'\,(\hat{\mathrm{s}}\cdot\hat{\mathrm{s}}')A(\hat{\mathrm{s}}\cdot\hat{\mathrm{s}}').$$

7 Analytic Continuation of the Wigner d-Functions

$$d^{\ell}_{mM}(\theta) \to d^{\ell}_{mM}[i\tau(x)]$$

using

$$\cos[i\tau(x)] = \sqrt{1+x^2}, \quad \sin[i\tau(x)] = ix.$$

Some examples:

$$d^0_{00} = 1,$$

$$\left\{ \begin{array}{ccc} d^1_{00} & = & \sqrt{1+x^2}, \\ d^1_{01} = -d^1_{10} & = & \frac{i}{\sqrt{2}}|x|, \\ d^1_{11} & = & \frac{1}{2}\left(\sqrt{1+x^2}+1\right), \end{array} \right.$$

8 Coefficients

We substitute

$$I(\mathbf{r},\hat{\mathrm{s}}) = \sum_{\mu=(M,n)}{}' \int \frac{\mathrm{d}^2 q}{(2\pi)^2} F_\mu(\mathbf{q}) I^{(+)}_{\mathbf{q}\mu}(\mathbf{r},\hat{\mathrm{s}}),$$

where

$$\begin{aligned} I^{(+)}_{\mathbf{q}\mu}(\mathbf{r},\hat{\mathrm{s}}) &= \mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{\rho}-Q_\mu(q)z} \sum_{\ell=|M|}^{\ell_{\mathrm{max}}} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathrm{s}}) \frac{\mathrm{e}^{-im\phi_{\mathbf{q}}}}{\sqrt{\sigma_\ell}} \\ &\times \langle \ell | \phi_n(M) \rangle \, d^{\ell}_{mM}[i\tau(q\lambda_\mu)], \end{aligned}$$

for

$$I(\boldsymbol{\rho}, z=0, \hat{\mathrm{s}}) = \delta(\boldsymbol{\rho}-\boldsymbol{\rho}_0)\,\delta(\hat{\mathrm{s}}-\hat{\mathrm{s}}_0) \quad [\hat{\mathrm{s}}\cdot\hat{z} > 0]\,.$$

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And multiply

$$\int_{\hat{\mathbf{s}}\cdot\hat{z}>0} \mathrm{d}^2 s\, Y^*_{\ell' m'}(\hat{\mathbf{s}}) \int \mathrm{d}^2\rho\, \mathrm{e}^{-i\mathbf{q}\cdot\boldsymbol{\rho}}.$$

9 Half-Range Integral

$$\int_{\hat{\mathbf{s}}\cdot\hat{z}>0} \mathrm{d}^2 s\, Y^*_{\ell' m'}(\hat{\mathbf{s}}) Y^*_{\ell m}(\hat{\mathbf{s}}) = \delta_{mm'} I^m_{\ell\ell'},$$

where

$$\begin{aligned} I^m_{\ell\ell'} &\equiv \frac{1}{2}\sqrt{\frac{(2\ell+1)(2\ell'+1)(\ell-m)!(\ell'-m)!}{(\ell+m)!(\ell'+m)!}} \\ &\times \int_0^1 P_{\ell m}(x) P_{\ell' m}(x) \mathrm{d}x. \end{aligned}$$

Properties of $I^m_{\ell\ell'}$:

- $I^m_{\ell\ell'} = \frac{1}{2}\delta_{\ell\ell'}$ if $\ell = \ell' =$ even or $\ell = \ell' =$ odd.
- $I^{-m}_{\ell\ell'} = I^m_{\ell\ell'}$.

10 Coef£cient Matrix

$$\sum_{\mu_2} A_{\mu_1,\mu_2}(q) f_{\mu_2}(q) = v_{\mu_1},$$

$$\Updownarrow$$

$$\sum_{Mn} A_{\ell' m', Mn}(q) f_{Mn}(q) = v_{\ell' m'},$$

where

$$\begin{aligned} A_{\ell' m', Mn}(q) &= \sum_{\ell=\max(|m'|,|M|)}^{\ell_{\max}} I_{\ell\ell'}^{m'} \frac{\langle \ell | \phi_n(M) \rangle}{\sqrt{\sigma_\ell}} d_{m'M}^{\ell} \left[i\tau(q\lambda_{Mn}) \right], \\ f_{Mn}(q) &= F_\mu(\mathbf{q}) \mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{\rho}_0}, \\ v_{\ell' m'} &= \delta_{m'0} \sqrt{\frac{2\ell'+1}{4\pi}}. \end{aligned}$$

11 Symmetry in the Linear Equations

$$\text{"Equation for } (\ell', -m')\text{"} = \text{"Equation for } (\ell', m')\text{"}$$

because

$$I_{\ell\ell'}^{-m'} = I_{\ell\ell'}^{m'},$$

$$d^{\ell}_{-m'M}\left[i\tau(q\lambda_{Mn})\right] = (-1)^{m'+M} d^{\ell}_{m'-M}\left[i\tau(q\lambda_{Mn})\right].$$

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Therefore

$$f_{-Mn}(q) = (-1)^{M} f_{Mn}(q).$$

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We consider

$$0 \le m' \le \ell_{\mathsf{max}}, \quad 0 \le M \le \ell_{\mathsf{max}}.$$

12 Two Sets of Parameters 1

$$\mu_1 = (\ell', m') \in \{(\ell', m') | 0 \le m' \le \ell',\ 0 \le \ell' \le \ell_{\mathsf{max}}\},$$

$$\mu_2 = (M, n) \in \{\mu | \lambda_\mu > 0,\ 1 \le n,\ 0 \le M \le \ell_{\mathsf{max}}\}$$

Choose $\mu_1 = (\ell', m')$ as

μ_1	ℓ'	m'
1	0	0
2	1	0
3	1	1
4	2	0
5	2	1
6	2	2
⋮	⋮	⋮

Give ℓ_{max} $\rightarrow$ Maximum number of equations, μ_{max}:

$$\mu_{\mathsf{max}} = \frac{(\ell_{\mathsf{max}} + 1)(\ell_{\mathsf{max}} + 2)}{2}.$$

13 Two Sets of Parameters 2

$$\mu_1 = (\ell', m') \in \{(\ell', m') | 0 \le m' \le \ell', \, 0 \le \ell' \le \ell_{\text{max}}\},$$
$$\mu_2 = (M, n) \in \{\mu | \lambda_\mu > 0, \, 1 \le n, \, 0 \le M \le \ell_{\text{max}}\}$$

Choose $\mu_2 = (M, n)$ as

μ_2	M	n
1	0	1
2	1	1
3	0	2
4	2	1
5	1	2
6	0	3
$\vdots$	$\vdots$	$\vdots$

Larger eigenvalues are more important:

$$\lambda_{\mu_2} > \lambda_{\mu_2+1}.$$

14 A Naive Way (Doesn't Work)

For $\ell_{\mathsf{max}} = 2$, make the 6×6 matrix A as

μ_1	ℓ'	m'	μ_2	M	n
1	0	0	1	0	1
2	1	0	2	1	1
3	1	1	3	0	2
4	2	0	4	2	1
5	2	1	5	1	2
6	2	2	6	0	3

$$\Longrightarrow \det A \simeq 0.$$

Some equations are numerically dependent!

$$-2I^0_{01}[\text{1st Eq.}] + [\text{2nd Eq.}] \simeq 2I^0_{12}[\text{4th Eq.}]$$

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6	2	2	6	0	3

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e.g.,

$$2I^0_{01}A_{11} - A_{21} + 2I^0_{12}A_{41} = \left[2(I^0_{01})^2 - \frac{1}{2} + 2(I^0_{12})^2\right] \frac{\langle 1|\phi_1(0)\rangle}{\sqrt{\sigma_1}} \sqrt{1 + (q\lambda_1)^2}$$

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$$\underbrace{\qquad\qquad}_{-0.0078125}$$

15 A Better Way

Strategy:

Collect $\bar{\mu}$ numerically independent equations making use of singular values s_j.

$$\bar{\mu} < \mu_{\mathsf{max}}.$$

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Example for $\ell_{\mathsf{max}} = 1$

1×1 **matrix**

$$A = A_{11}.$$

2×2 matrix Consider

$$A = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}.$$

←

2×2 matrix Consider

$$A = \begin{pmatrix} A_{11} & A_{12} \end{pmatrix}.$$

Add the next row:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

2×2 matrix Consider

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Add the next row:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Here the singular values are

$$s_1 = 9.0099079, \quad s_2 = 0.$$

2×2 matrix Consider

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Here the singular values are

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Discard this row and try the next row:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{pmatrix}.$$

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$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{pmatrix}.$$

Here the singular values are

$$s_1 = 6.7669220, \quad s_2 = 0.2833575.$$

3×3 matrix Consider

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \end{pmatrix}.$$

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$$s_1 = 9.0125599, \quad s_2 = 0.0015772.$$

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3×3 matrix Consider

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$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}.$$

Here the singular values are

$$s_1 = 9.0125599, \quad s_2 = 0.2833575, \quad s_3 = 0.0015772.$$

$$\rightarrow \bar{\mu} = 3.$$

For $\ell_{\mathsf{max}} = 2$, $\bar{\mu} = 5$, the matrix A is a 5×5 matrix:

μ_1	ℓ'	m'	μ_2	M	n
1	0	0	1	0	1
2	1	0	2	1	1
3	1	1	3	0	2
	2	0	4	2	1
4	2	1	5	1	2
5	2	2		0	3

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μ_1	ℓ'	m'	μ_2	M	n
1	0	0	1	0	1
2	1	0	2	1	1
3	1	1	3	0	2
	2	0	4	2	1
4	2	1	5	1	2
5	2	2		0	3

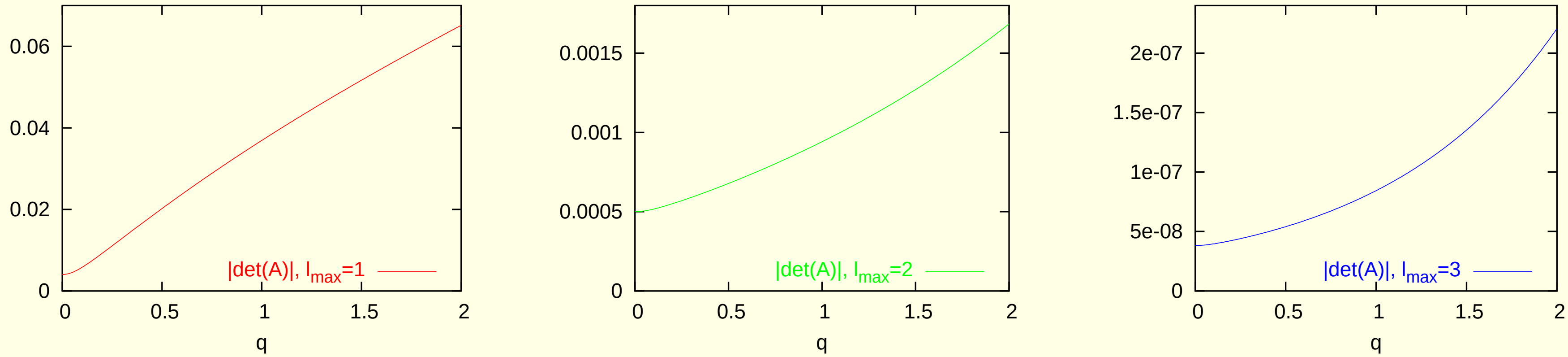
$$\ell_{\mathsf{max}} = 5 \rightarrow \mu_{\mathsf{max}} = 21 \Longrightarrow \bar{\mu} = 15$$

$$\ell_{\mathsf{max}} = 10 \rightarrow \mu_{\mathsf{max}} = 66 \Longrightarrow \bar{\mu} = 26$$

$$\ell_{\mathsf{max}} = 15 \rightarrow \mu_{\mathsf{max}} = 136 \Longrightarrow \bar{\mu} = 36$$

$$\ell_{\mathsf{max}} = 20 \rightarrow \mu_{\mathsf{max}} = 231 \Longrightarrow \bar{\mu} = 43$$

16 Determinant as a Function of q



$$\mu_a/\mu_s = 6.0 \times 10^{-5}, \quad g = 0.98.$$

17 Intensity for the Half Space

Let us put

$$\widehat{\mathbf{s}} = (\theta_{\mathbf{s}}, \phi_{\mathbf{s}}) = (\theta_{\mathbf{s}}, 0).$$

We obtain

$$I(\mathbf{r}, \widehat{\mathbf{s}}) = \frac{1}{2\pi} \sum_{\ell m} \frac{i^m}{\sqrt{\sigma_\ell}} Y_{\ell m}(\widehat{\mathbf{s}}) K_{\ell m},$$

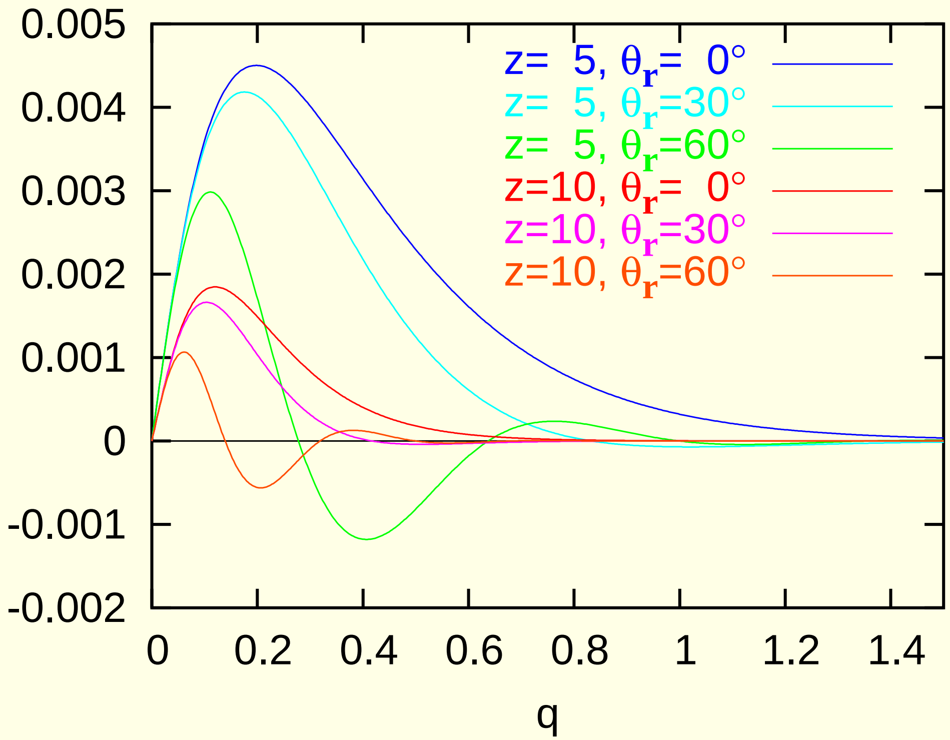
where

$$\begin{aligned} K_{\ell m} &= \int_0^{q_{\mathrm{max}}} \mathrm{d}q\, q J_m(q|\boldsymbol{\rho} - \boldsymbol{\rho}_0|) \sum_{M \geq 0, n} (2 - \delta_{M0}) \mathrm{e}^{-Q_{Mn}(q) z} f_{Mn}(q) \\ &\times \langle \ell | \phi_n(M) \rangle\, d^{\ell}_{mM}[i\tau(q\lambda_\mu)]. \end{aligned}$$

18 Integrand in the Kernel

Integrand in $K_{\ell=0,m=0}$:

$$q J_m(q|\boldsymbol{\rho}-\boldsymbol{\rho}_0|)$$
$$\times \sum_{M\geq 0,n} (2-\delta_{M0}) \mathrm{e}^{-Q_{Mn}(q)z} f_{Mn}(q) \langle \ell | \phi_n(M) \rangle \, d^{\ell}_{mM}[i\tau(q\lambda_\mu)].$$

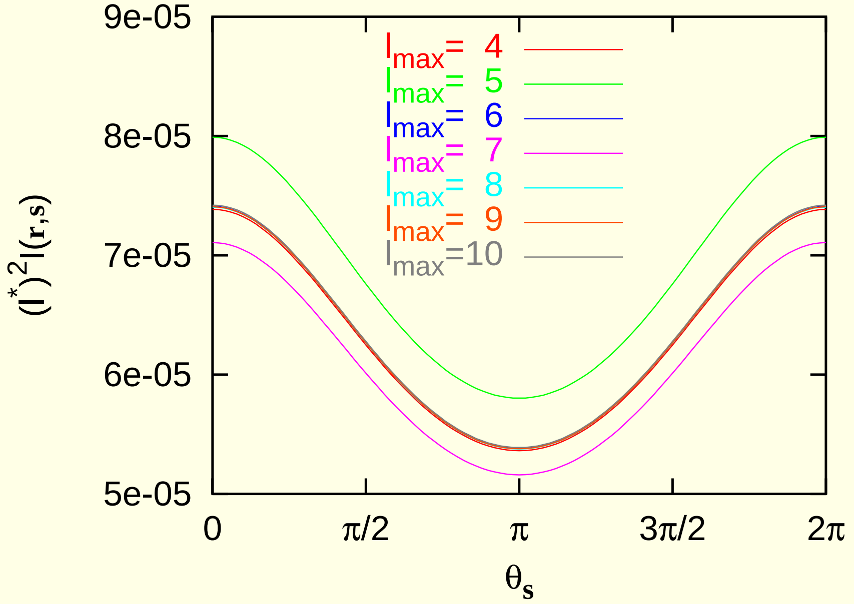


$(\ell_{\max} = 10,\ \mu_{\max} = 66,\ \bar{\mu} = 26)$

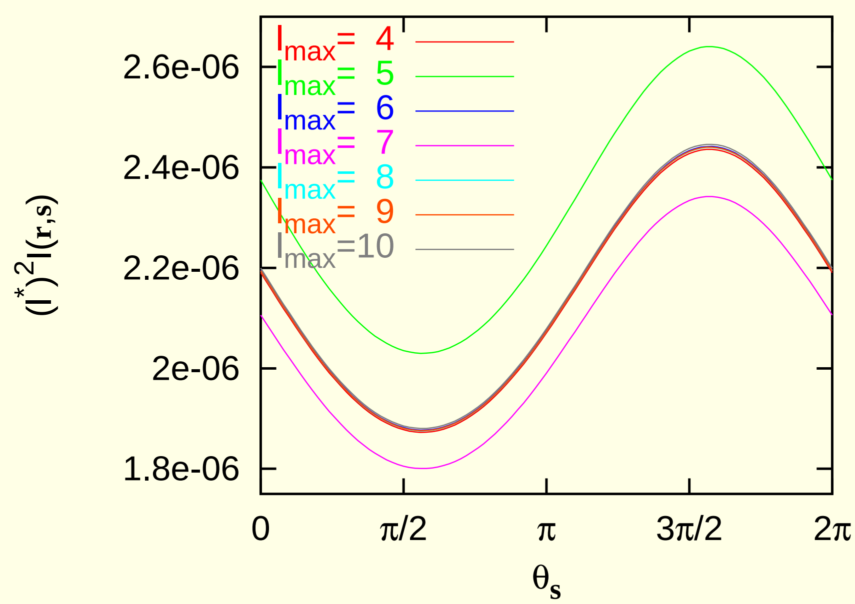
19 Convergence of Intensity 1

$z = 20\ell^*$

$\theta_{\mathbf{r}} = 0$



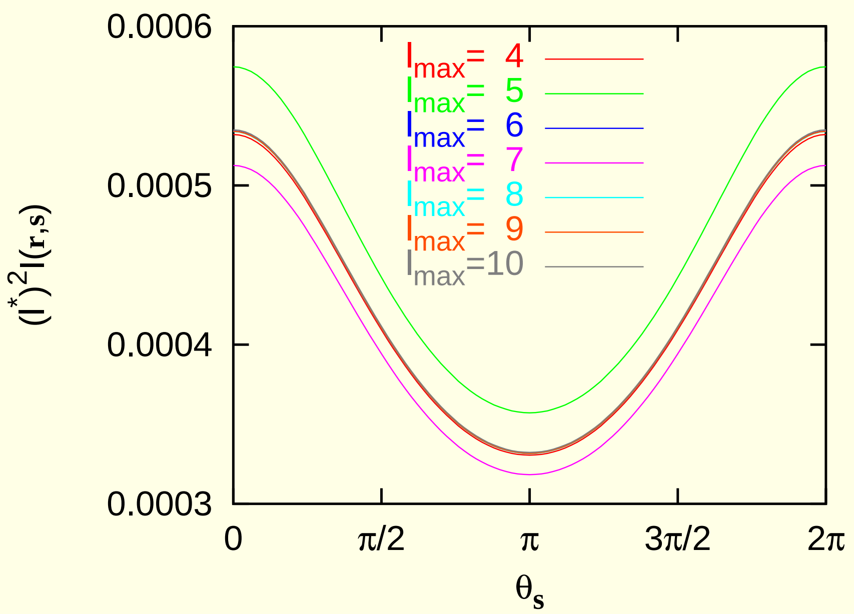
$\theta_{\mathbf{r}} = \frac{\pi}{3}$



20 Convergence of Intensity 2

$z = 10\ell^*$

$\theta_{\mathbf{r}} = 0$



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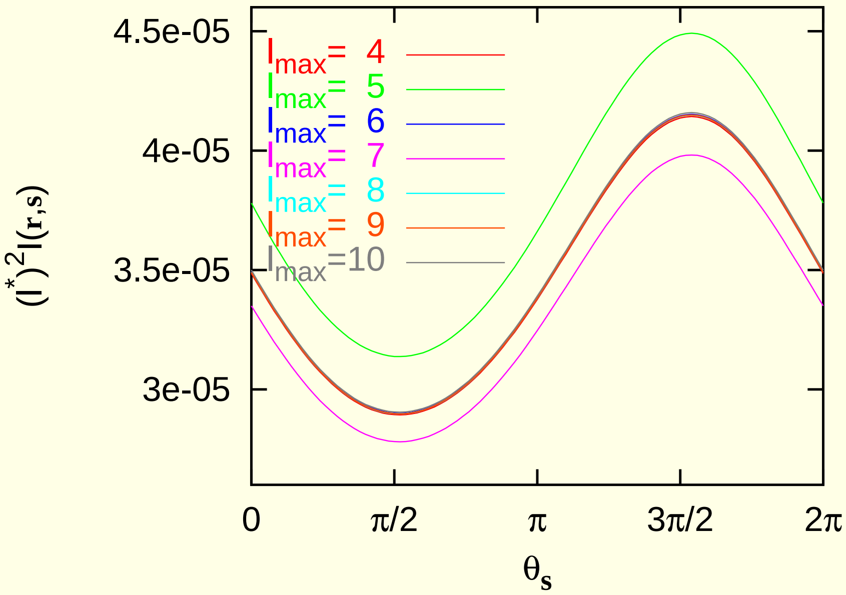
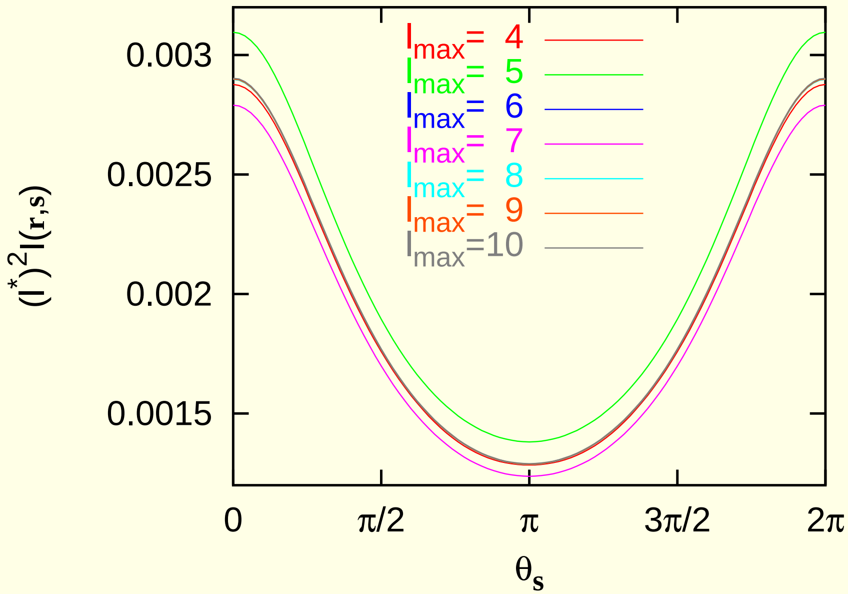


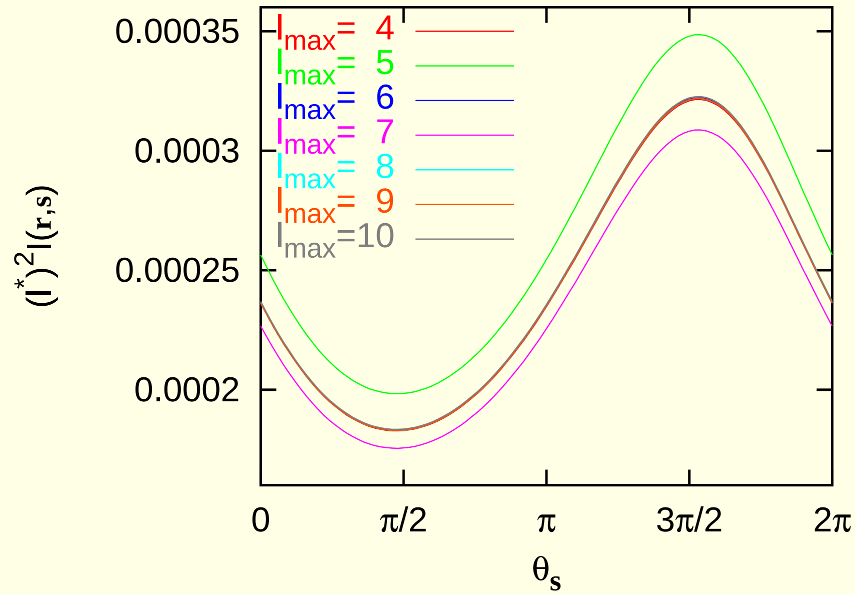
21 Convergence of Intensity 3

$z = 5\ell^*$

$\theta_{\mathbf{r}} = 0$



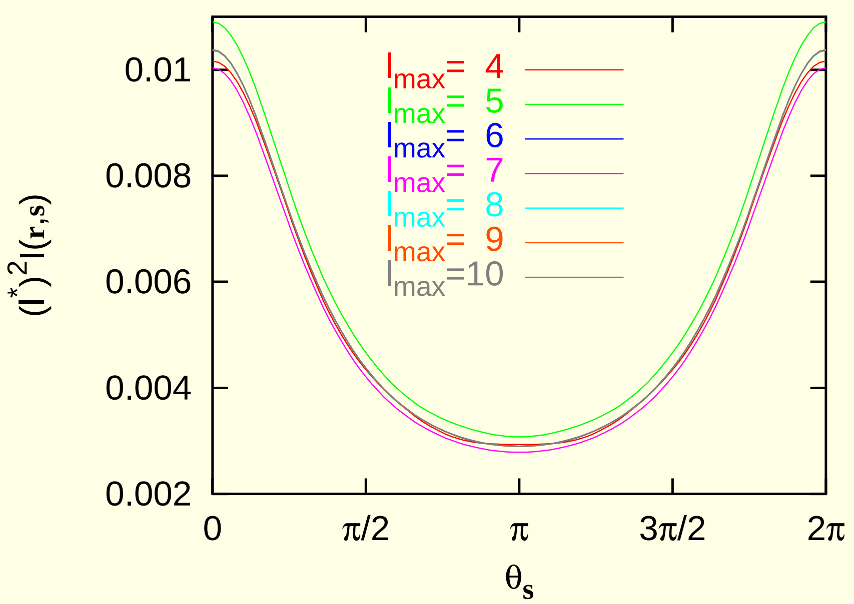
$\theta_{\mathbf{r}} = \frac{\pi}{3}$



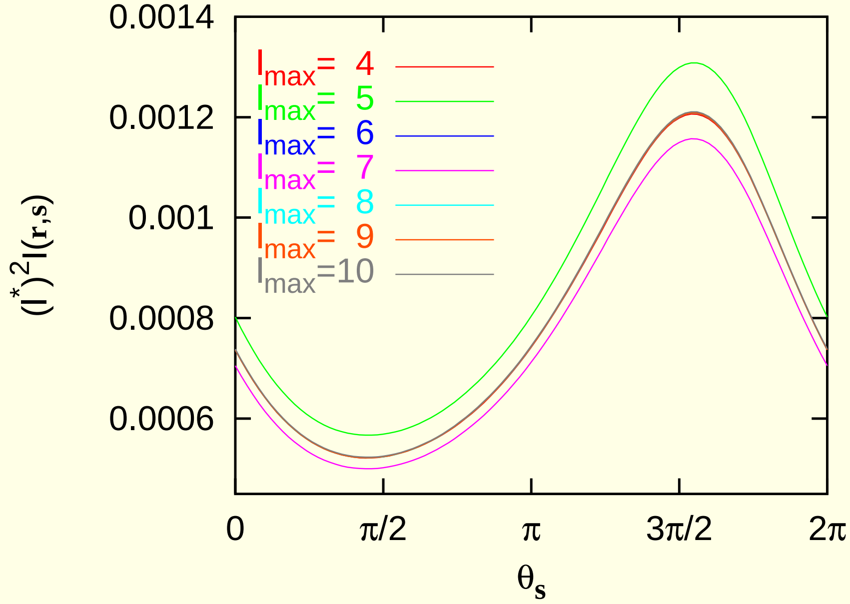
22 Convergence of Intensity 4

$z = 3\ell^*$

$\theta_\mathbf{r} = 0$



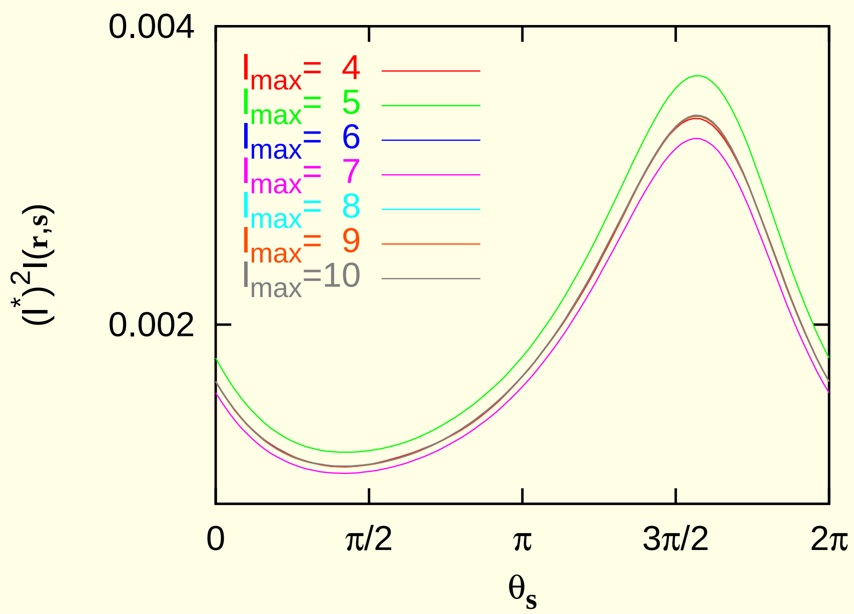
$\theta_\mathbf{r} = \frac{\pi}{3}$



23 Convergence of Intensity 5

$z = 2\ell^*$

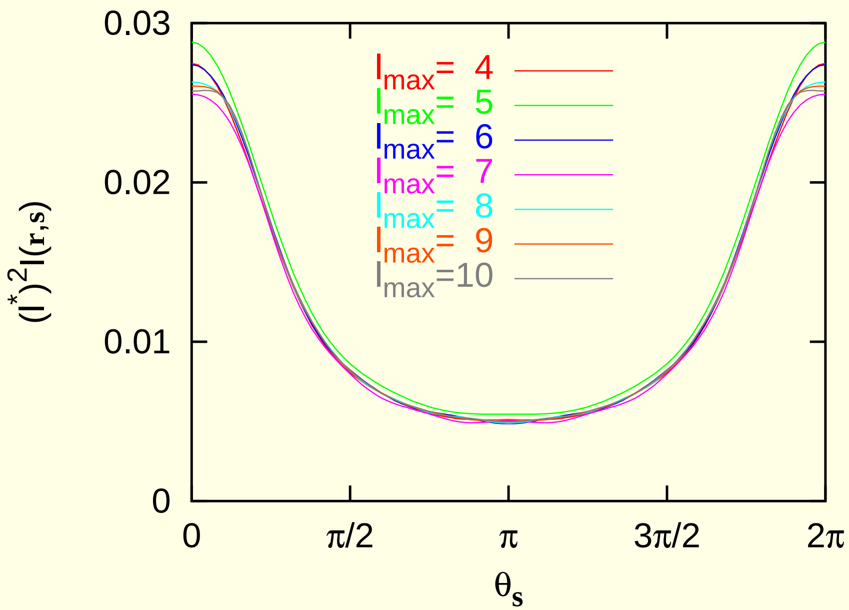
$$\theta_{\mathbf{r}} = \frac{\pi}{3}$$

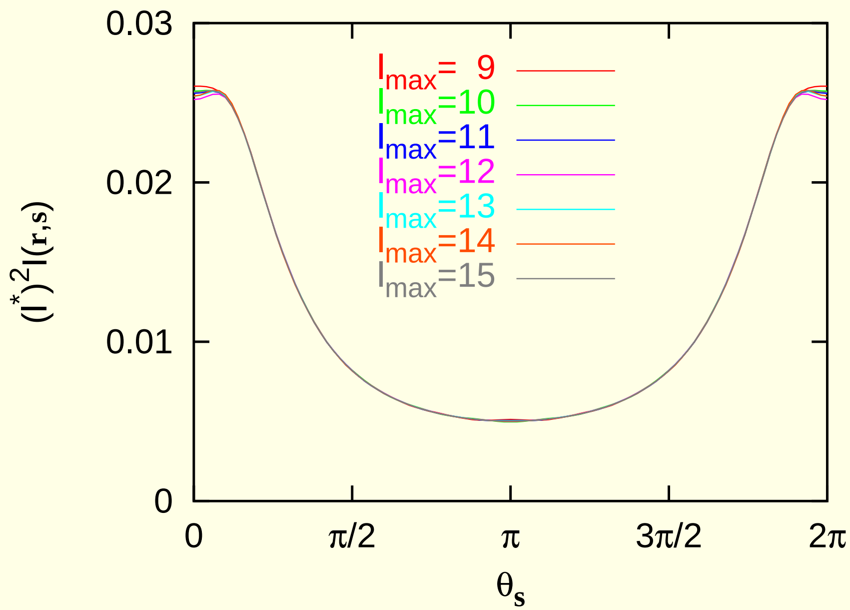


24 Convergence of Intensity 6

$z = 2\ell^*$

$\theta_{\mathbf{r}} = 0$

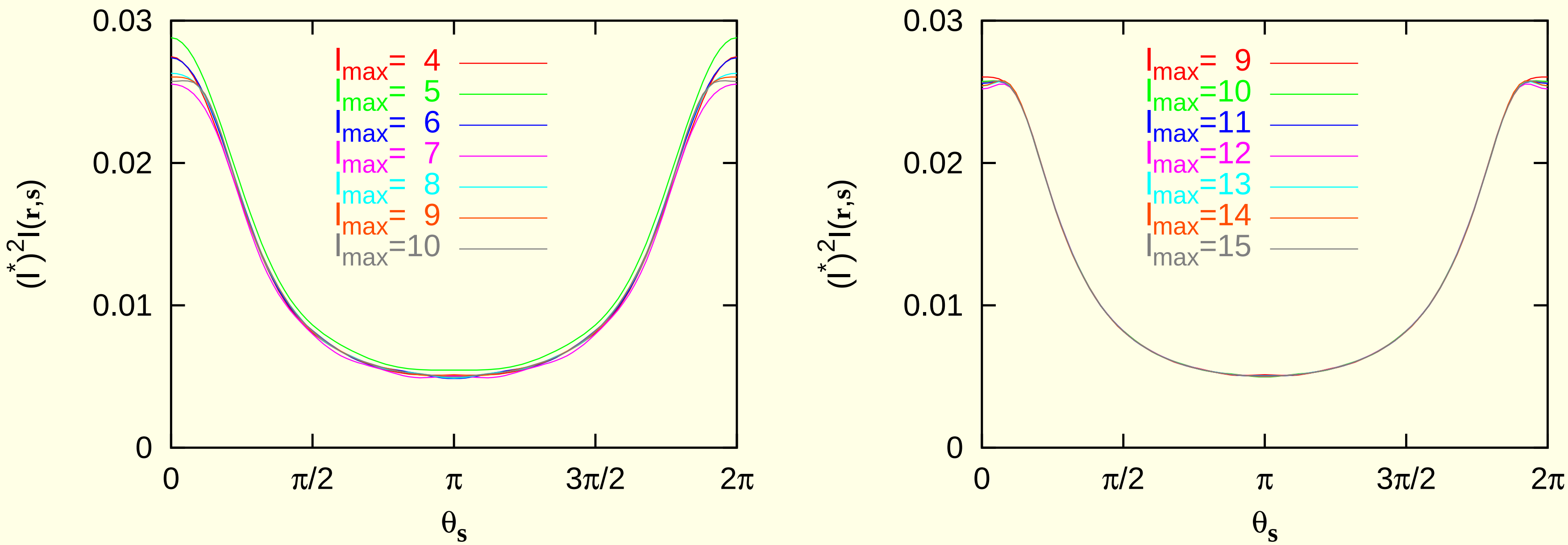




24 Convergence of Intensity 6

$z = 2\ell^*$

$\theta_{\mathbf{r}} = 0$



Effect of Ballistic Transport!

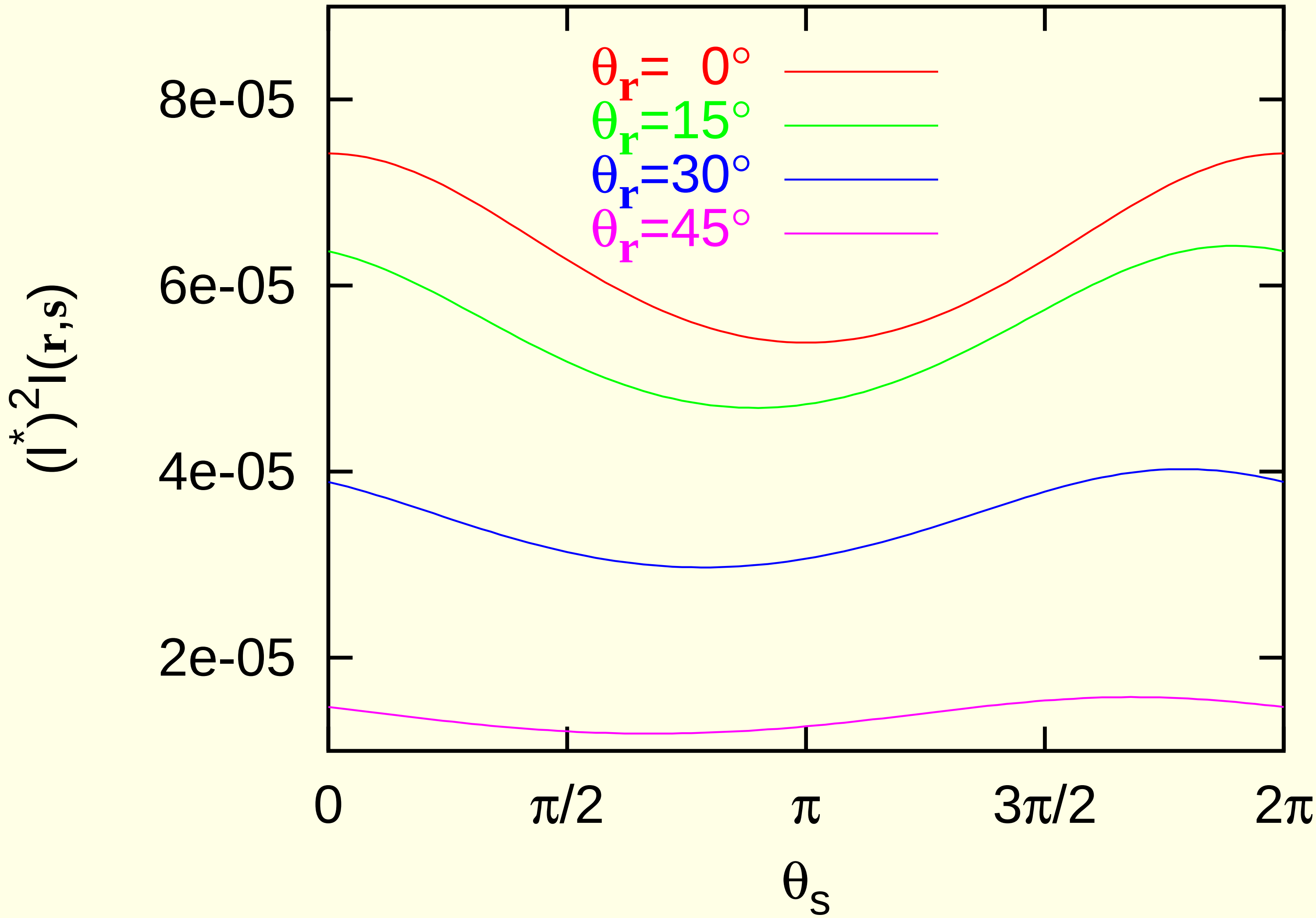
25 Summary

I am closing my talk by showing five more figures.

- Angular dependence of the intensity at $z = 20\ell^*$.
- Angular dependence of the intensity at $z = 10\ell^*$.
- Angular dependence of the intensity at $z = 5\ell^*$.
- Angular dependence of the intensity at $z = 3\ell^*$.
- Angular dependence of the intensity at $z = 2\ell^*$ (effect of ballistic transport).

26 Figure 1

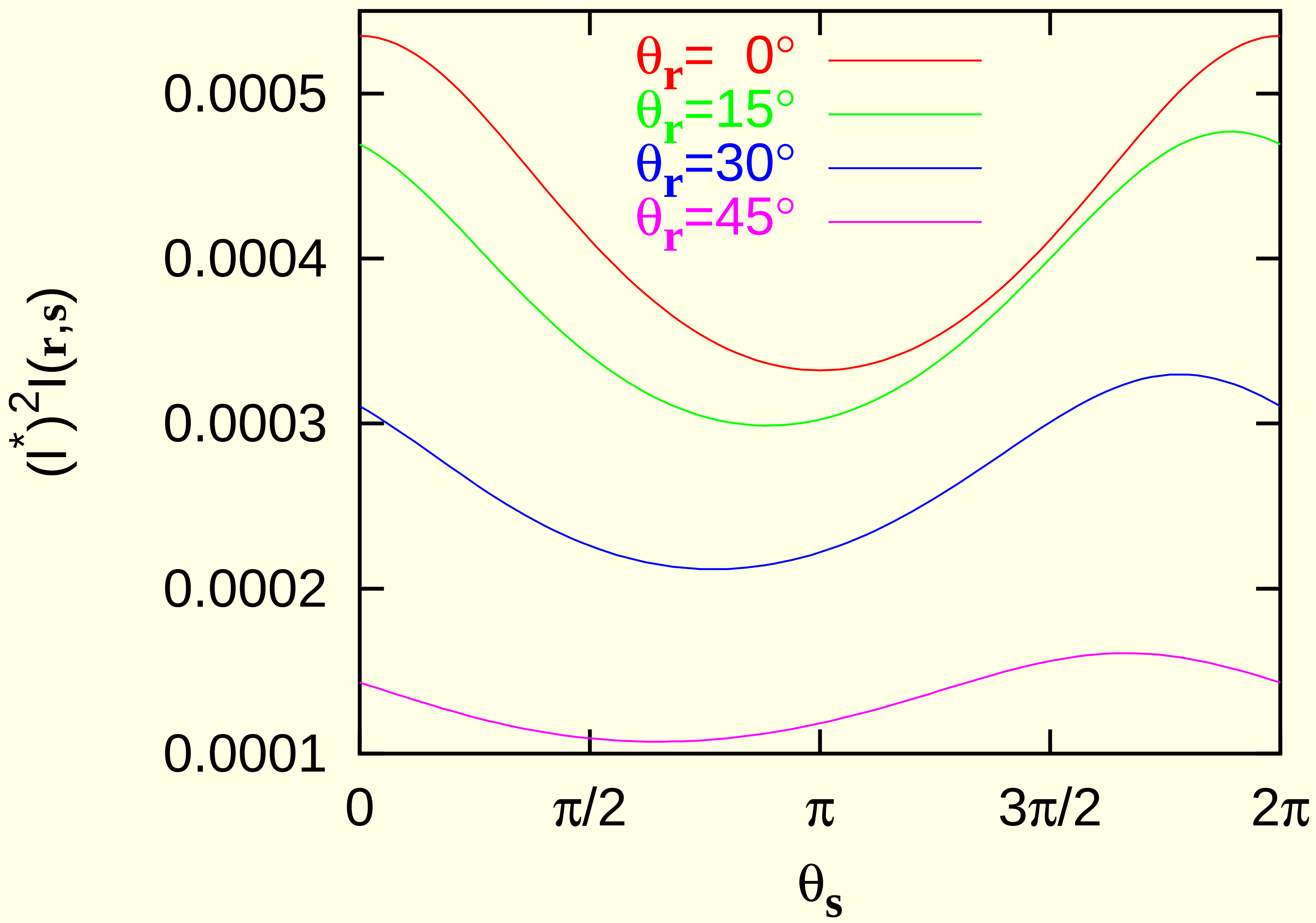
$z = 20\ell^*$



$\ell_{\max} = 10$

27 Figure 2

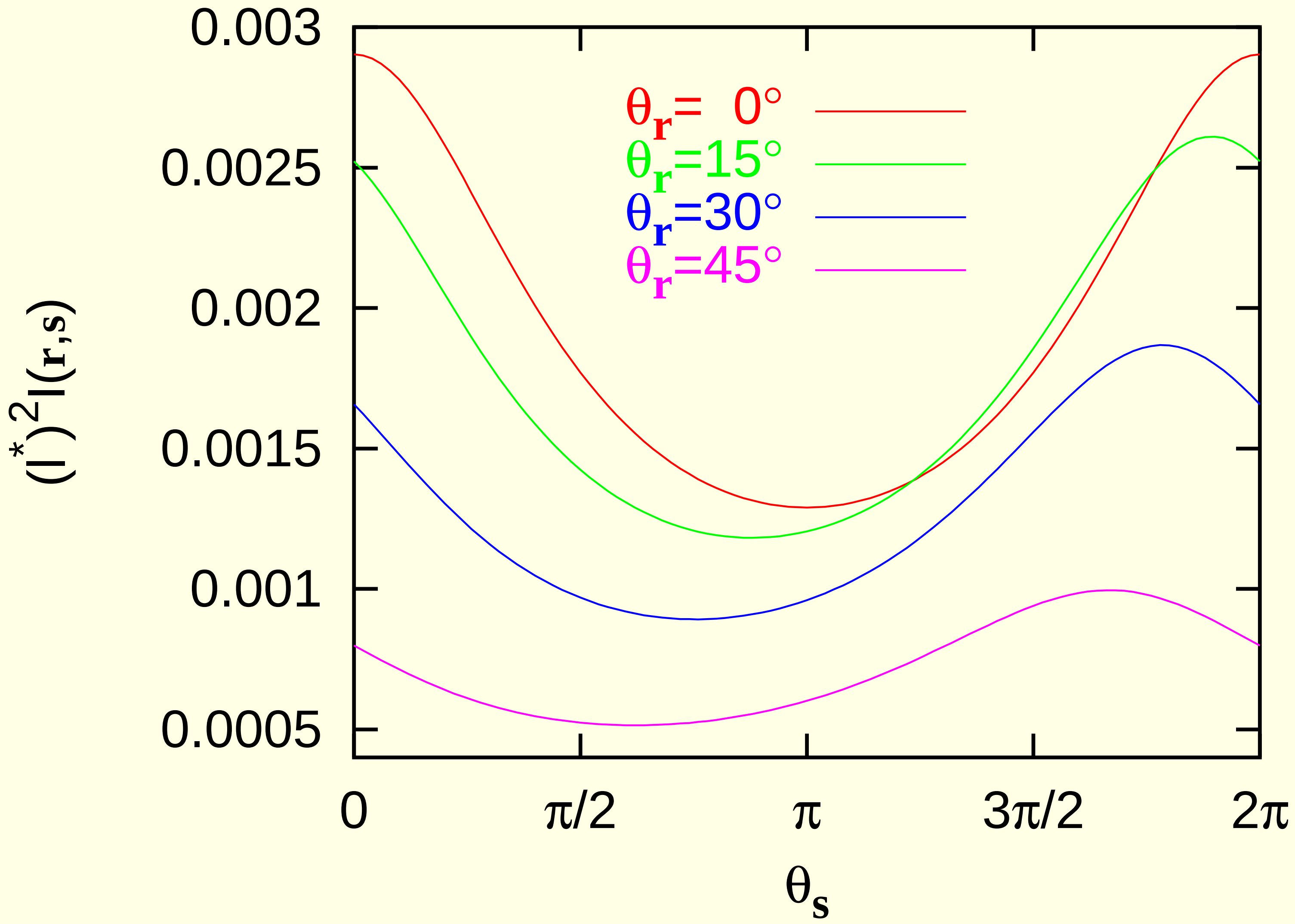
$z = 10\ell^*$



$\ell_{\max} = 10$

28 Figure 3

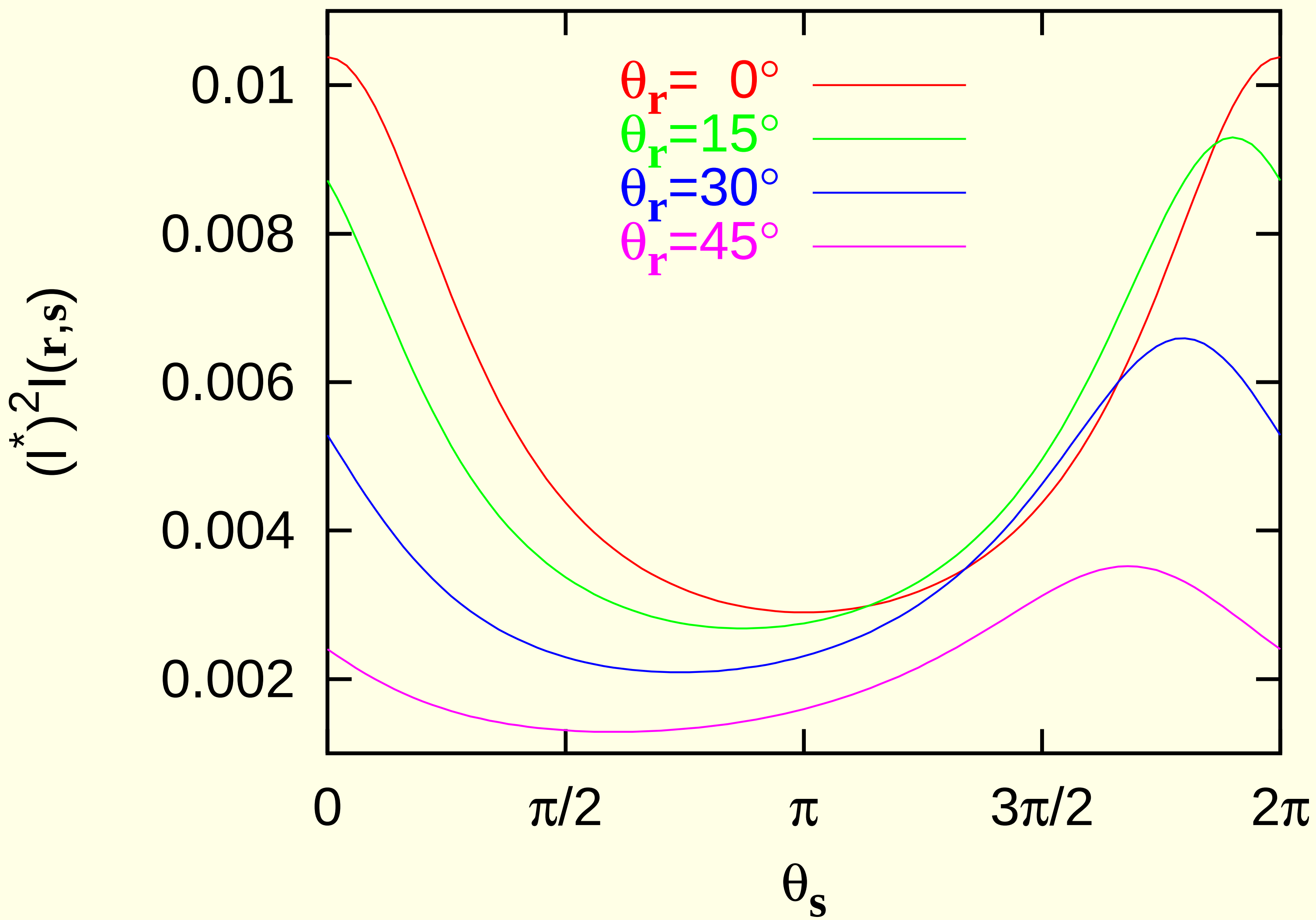
$z = 5\ell^*$



$\ell_{\mathrm{max}} = 10$

29 Figure 4

$z = 3\ell^*$



$\ell_{\mathrm{max}} = 10$

30 Figure 5

$z = 2\ell^*$ and $\theta_{\mathbf{r}} = 0$